Math 261 — Exercise sheet 9

http://staff.aub.edu.lb/~nm116/teaching/2018/math261/index.html

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The use of calculators is recommended, at least for the first exercise.

Exceptionally, none of these exercises are mandatory.

Exercise 9.1: Continued fraction expansions

1. Express the rational $\dfrac{2018}{261}$ as a continued fraction.

2. Compute the continued fraction expansion of $\dfrac{e+1}{e-1}$ where

$$e = \exp(1) \approx 2.718281828.$$

   Can you see a pattern? Also compute the 4 first convergents, and see how many correct decimals they have compared to $\frac{e+1}{e-1}$.

3. Assuming that the pattern you spotted in the previous question does hold, prove that e is irrational.

Solution 9.1:

1. We apply the process seen in class. Since $\frac{2018}{261} \in \mathbb{Q}$, it will stop after finitely many steps, and we will get a representation of $\frac{2018}{261}$ as a continued fraction.

$$\begin{aligned}
x_0 &= \frac{2018}{261}, & a_0 &= \lfloor x_0 \rfloor = 7,\\
x_1 &= \frac{1}{\frac{2018}{261}-7} = \frac{261}{191}, & a_1 &= \lfloor x_1 \rfloor = 1,\\
x_2 &= \frac{1}{\frac{261}{191}-1} = \frac{191}{70}, & a_2 &= \lfloor x_2 \rfloor = 2,\\
x_3 &= \frac{1}{\frac{191}{70}-2} = \frac{70}{51}, & a_3 &= \lfloor x_3 \rfloor = 1,\\
x_4 &= \frac{1}{\frac{70}{51}-1} = \frac{51}{19}, & a_4 &= \lfloor x_4 \rfloor = 2,\\
x_5 &= \frac{1}{\frac{51}{19}-2} = \frac{19}{13}, & a_5 &= \lfloor x_5 \rfloor = 1,\\
x_6 &= \frac{1}{\frac{19}{13}-1} = \frac{13}{6}, & a_6 &= \lfloor x_6 \rfloor = 2,\\
x_7 &= \frac{1}{\frac{13}{6}-2} = 6, & a_7 &= \lfloor x_7 \rfloor = 6.
\end{aligned}$$

Since $x_7 = a_7$ is an integer, we stop there. We have found that

$$\frac{2018}{261} = [7,1,2,1,2,1,2,6] = 7 + \cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{6}}}}}}}.$$

2. We start with $x_0 = \frac{e+1}{e-1} = 2.163\cdots$. We then find that $a_0 = 2$, so $x_1 = \frac{1}{2.163\cdots-2} = 6.099\cdots$, so $a_1 = 6$, so $x_2 = \frac{1}{6.099\cdots-6} = 10.0711\cdots$, so $a_2 = 10$, so $x_3 = \frac{1}{10.0711\cdots-10} = 14.055\cdots$, so $a_3 = 14$, so $x_4 = \frac{1}{14.055\cdots-14} = 18.045\cdots$, so $a_4 = 18$, and so on. It seems that $a_n = 4n+2$ for all n...

Anyway, we can compute the first values of p_n and q_n:

n	-1	0	1	2	3	$\cdots$	
a_n	$\bullet$	2	6	10	14	18	$\cdots$
p_n	1	2	13	132	1861	$\cdots$	
q_n	0	1	6	61	860	$\cdots$	

The first 4 convergents are thus 2 (0 correct decimals), 13/6 (2 correct decimals), 132/61 (4 correct decimals), and 1861/860 (7 correct decimals!).

3. Suppose by contradiction that e were rational. Then $\dfrac{e+1}{e-1}$ would also be rational, so its continued fraction expansion would terminate, which contradicts our guess that $a_n = 4n+2$ for all n.

   Euler proved that indeed $a_n = 4n+2$ for all n, and used this to prove that e is irrational. The number e is named after him.

Exercise 9.2: A Pell-Fermat equation

1. Compute the continued fraction expansion of $\sqrt{6}$.

2. Use the previous question to find the fundamental solution to the equation $x^2 - 6y^2 = 1$.

3. Use the ring structure of $\mathbb{Z}[\sqrt{6}]$ to find 2 other non-trivial solutions (changing the signs of x and y does not count !)

Solution 9.2:

1. Let $x = \sqrt{6}$. The continued fraction expansion starts as follows:

$$\begin{aligned} x_0 &= \sqrt{6}, & a_0 &= \lfloor x_0 \rfloor = 2, & p_0 &= 2, q_0 = 1, \\ x_1 = \frac{1}{\sqrt{6}-2} &= \frac{2+\sqrt{6}}{2}, & a_1 &= \lfloor x_1 \rfloor = 2, & p_1 &= 5, q_1 = 2, \\ x_2 = \frac{1}{\frac{2+\sqrt{6}}{2}-2} &= 2+\sqrt{6}, & a_2 &= \lfloor x_2 \rfloor = 4, & p_2 &= 22, q_2 = 9, \\ x_3 = \frac{1}{2+\sqrt{6}-4} &= \frac{2+\sqrt{6}}{2}, & & \cdots \end{aligned}$$

   Since $x_3 = x_1$, the process becomes periodic from this point on. We deduce that

$$\sqrt{6} = [2, \overline{2, 4}].$$

2. We compute the first few values of p_n and q_n, until $p_n^2 - 6q_n^2 = \pm 1$.

n	0	1	$\cdots$
a_n	2	2	$\cdots$
p_n	2	5	$\cdots$
q_n	1	2	$\cdots$
$p_n^2 - 6q_n^2$	-2	1	$\cdots$

   Luckily we don't have to go very far ! We find the solution $x = 5$, $y = 2$.

3. We have thus found the element $\alpha = 5 + 2\sqrt{6} \in \mathbb{Z}[\sqrt{6}]$ of norm $N(\alpha) = 1$. Since the norm is multiplicative, all the powers of α also have norm 1, so correspond to solutions of the equation $x^2 - 6y^2 = 1$.

We compute $\alpha^2 = 49 + 20\sqrt{6}$, whence the solution $x = 49$, $y = 20$.

Also, $\alpha^3 = 485 + 198\sqrt{6}$, whence the solution $x = 485$, $y = 198$.

Of course, we could go on if we wanted to !

Exercise 9.3: The battle of Hastings

The battle of Hastings was a major battle in English history. It took place on October 14, 1066.

The following fictional historical text, taken from *Amusement in Mathematics* (H. E. Dundeney, 1917), refers to it:

"The men of Harold stood well together, as their wont was, and formed thirteen squares, with a like number of men in every square thereof. (...) When Harold threw himself into the fray the Saxons were one mighty square of men, shouting the battle cries 'Ut!', 'Olicrosse!', 'Godemite!'."

Use continued fractions to determine how many soldiers this fictional historical text suggests Harold II had at the battle of Hastings.

Solution 9.3:

We are looking for solutions to $13y^2 + 1 = x^2$ with some $x, y \in \mathbb{N}$. This translates into the Pell-Fermat equation $x^2 - 13y^2 = 1$.

Clearly, the trivial solution $x = 1$, $y = 0$ does not reflect the situation (I doubt Harold II would have gone to battle alone !), so let us compute the continued fraction expansion of $x = \sqrt{13}$ until we find a non-trivial solution.

$$x_0 = \sqrt{13}, \quad a_0 = \lfloor x_0 \rfloor = 3, \quad p_0 = 3, q_0 = 1, \quad p_0^2 - 13q_0^2 = -4 \neq \pm 1.$$

$$x_1 = \frac{1}{\sqrt{13} - 3} = \frac{3 + \sqrt{13}}{4}, \quad a_1 = \lfloor x_1 \rfloor = 1, \quad p_1 = 4, q_1 = 1, \quad p_1^2 - 13q_1^2 = 3 \neq \pm 1.$$

$$x_2 = \frac{1}{\frac{3+\sqrt{13}}{4} - 1} = \frac{1 + \sqrt{13}}{3}, \quad a_2 = \lfloor x_2 \rfloor = 1, \quad p_2 = 7, q_2 = 2, \quad p_2^2 - 13q_2^2 = -3 \neq \pm 1.$$

$$x_3 = \frac{1}{\frac{1+\sqrt{13}}{3} - 1} = \frac{2 + \sqrt{13}}{3}, \quad a_3 = \lfloor x_3 \rfloor = 1, \quad p_3 = 11, q_3 = 3, \quad p_3^2 - 13q_3^2 = 4 \neq \pm 1.$$

$$x_4 = \frac{1}{\frac{2+\sqrt{13}}{3} - 1} = \frac{1 + \sqrt{13}}{4}, \quad a_4 = \lfloor x_4 \rfloor = 1, \quad p_4 = 18, q_4 = 5, \quad p_4^2 - 13q_4^2 = -1.$$

We have found the element $\alpha = 18 + 5\sqrt{13}$ of norm $N(\alpha) = -1$. We deduce that the fundamental solution to our equation corresponds to

$$\alpha^2 = 649 + 180\sqrt{13},$$

that is to say $x = 649$, $y = 180$.

Since the other solutions are even larger, this suggests a number of soldiers on this side of the battle (including Harold II) was at least $649^2 = 421201$. That's really a lot, which confirms that this text is certainly fictional!

Exercise 9.4: More Pell-Fermat

Redo exercise 9.2, with 6 replaced by 14, 15, 17, and 18.

Solution 9.4:

- For 14: We find $\sqrt{14} = [3, \overline{1, 2, 1, 6}]$; the fundamental solution to $x^2 - 14y^2$ is $x = 15$, $y = 4$.
- For 15: We find $\sqrt{15} = [3, \overline{1, 6}]$; the fundamental solution to $x^2 - 15y^2$ is $x = 4$, $y = 1$.
- For 17: We find $\sqrt{17} = [4, \overline{8}]$; the fundamental solution to $x^2 - 17y^2$ is $x = 33$, $y = 8$.
- For 18: We find $\sqrt{18} = [4, \overline{4, 8}]$; the fundamental solution to $x^2 - 18y^2$ is $x = 17$, $y = 4$.

Exercise 9.5: Continued fraction vs. series

Let $x \in (0, 1)$ be irrational, and let $[a_0, a_1, \cdots, a_n] = p_n/q_n$ ($n \in \mathbb{N}$) be the convergents of the continued fraction expansion of x. Prove that

$$x = \sum_{n=0}^{+\infty} \frac{(-1)^n}{q_n q_{n+1}}.$$

Hint: Where could the $(-1)^n$ come from ?

Solution 9.5:

We know that $q_n p_{n-1} - p_n q_{n-1} = (-1)^n$ for all n. Therefore, we have

$$\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^{n-1}}{q_n q_{n-1}}$$

for all n. Now, obviously

$$\frac{p_m}{q_m} = \left(\frac{p_m}{q_m} - \frac{p_{m-1}}{q_{m-1}}\right) + \left(\frac{p_{m-1}}{q_{m-1}} - \frac{p_{m-2}}{q_{m-2}}\right) + \cdots + \left(\frac{p_1}{q_1} - \frac{p_0}{q_0}\right) \frac{p_0}{q_0} = \frac{p_0}{q_0} + \sum_{n=1}^{m} \left(\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}\right).$$

On the one hand, $p_0 = a_0 = \lfloor x \rfloor = 0$ since $x \in (0, 1)$, so $\frac{p_0}{q_0} = 0$. On the other hand, we know that the sequence $\frac{p_n}{q_n}$ converges to x, so we get

$$x = \lim_{m\to\infty} \frac{p_m}{q_m} = \lim_{m\to\infty} \sum_{n=1}^{m} \left(\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}\right) = \sum_{n=1}^{+\infty} \left(\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}\right) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{q_n q_{n-1}} = \sum_{n=0}^{+\infty} \frac{(-1)^n}{q_n q_{n+1}}.$$