Gröbner Bases - Assignment 5

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05/04/2016

1 Problem 1

1.1 part (i)

Let $A = \mathbb{C}\left\langle x, y\right\rangle / I$ and $I = (f), f = ay^2 + byx + cxy + dx^2$. Since $a = b = c = 0$ and a, b, c, d not all simultaneously zero, $d \neq 0$ and so we can assume W.L.O.G. that $d = 1$. Thus $I = (x^2)$, which has reduced Gröbner basis G consisting of x^2 since the S-polynomial forming from the self overlap x of x^2 goes to 0. Let x, y be labelled 1,2. Since x, y are the reduced monomials of degree 1, we get the graph of reduced words Λ where (i, j) denotes a path exists from i to j:
$xx = xx,$ but this is not reduced w.r.t. G
$xy = xy, (1, 2) \in \Lambda$
$yx = yx, (2, 1) \in \Lambda$
$yy = yy, (2, 2) \in \Lambda$
So we get the incidence matrix M:

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

This has characteristic polynomial $p_M(t) = \det(t\mathbb{I} - M) = t^2 - t - 1$. Thus we get the recurrence relation

$$n_{2+j}(A) - n_{1+j}(A) - 1n_j(A) = 0, \forall j \geq 1$$

Since x, y are reduced w.r.t. G, $n_1(A) = 2$. $n_2(A) = 3$, which is the sum of the entries of M. $n_3(A) = 5$ by the recurrence realtion. The recurrence relation is interestingly the same as that of the Fibonacci numbers, and so the dimension of the n-th homogenous component of A is the $n+3$ Fibonacci number. We consider the Fibonacci numbers to be:

$$0, 1, 1, 2, 3, 5, 8, ...$$

1.2 part (ii)

In this case $c \neq 0$. Thus G the reduced Gröbner basis of I consists of one term with leading monomial xy since we have no overlaps. Keep the labelling x, y as 1,2. In this case we get the graph of reduced words Λ:
$(1,1) \in \Lambda$ since now x^2 is reduced w.r.t. G
xy is now non-reduced
$(2,1) \in \Lambda, (2,2) \in \Lambda$ as before
So we get the incidence matrix M:

$$M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

We get the characteristic polynomial $p_M(t) = (t-1)^2 = t^2 - 2t + 1$ and thus the recurrence relation

$$n_{2+j}(A) - 2n_{1+j}(A) + n_j(A) = 0, \forall j \geq 1$$

We have $n_1(A) = 2 = 1 + 1, n_2(A) = 3 = 2 + 1$ as in part (i). Thus our recurrence relation gives $n_3(A) = 4 = 3 + 1$. More generally $n_k(A) = 2n_{k-1}(A) - n_{k-2}(A) = 2(k) - (k-1) = k + 1$.

1.3 part (iii)

In this case $b \neq 0$. Thus G the reduced Gröbner basis of I consists of one term with leading monomial yx since we have no overlaps. In this case we get the graph of reduced words Λ:
$(1,1), (1,2), (2,2) \in \Lambda$
So we can easily see that the sum of entries in powers of M and the characteristic polynomial of our incidence matrix M will be the same as in case (ii). Thus $n_k(A) = k + 1$ as in case (ii).

2 Problem 2

We are considering $I = (f)$ where $f = ay^2 + byx + cxy + dx^2, a \neq 0$. So I contains no elements of degree 1, thus $\dim I_{(1)} = 0$.
I contains a single homogenous element of degree 2, namely f, thus $\dim I_{(2)} = 1$.
Now we can certainly span $I_{(3)}$ by the polynomials xf, yf, fx, fy, since that is all possible ways to multiply f by a monomial of degree 1. It may not form a basis however, and thus $\dim I_{(3)} \leq 4$.
To see the corresponding inequality for the general $\dim I_k$ note the following. To span $I_{(k)}$ we can put $k-2$ monomials of degree 1 around f, in the following manner: hfh' where h, h' are words consisting of x, y and $\deg(h) + \deg(h') = k - 2$. Since there is only two choices for each letter in h, h', when we fix the length of h we obtain 2^{k-2} homogenous elements of degree k. When we vary h, we have $k-1$ choices for the length of h and thus we get $\dim I_{(k)} \leq (k-1)2^{k-2}$.

3 Problem 3

For this question, we know that yf, fy, xf, fx span $I_{(3)}$ so if we can get one to be a linear combination of the others then we would guarantee $\dim I_{(3)} = 3$. So to do this let us set $-yf + e_1(fy) + e_2(xf) + e_3(fx) = 0,\ e_1, e_2, e_3 \in \mathbb{C}$. We get the following:

$$\begin{aligned}
& -y^3 - by^2x - cyxy - dyx^2 \\
& + e_1(y^3 + byxy + cxy^2 + dx^2y) \\
& + e_2(xy^2 + bxyx + cx^2y + dx^3) \\
& + e_3(y^2x + byx^2 + cxyx + dx^3) = 0
\end{aligned}$$

We can then find some equations for the unknowns by comparing co-efficients of like terms:
$-y^3 + e_1y^3 = 0 \implies e_1 = 1$
$y^2x : b = e_3,\ yxy : c = e_1b = b,\ yx^2 : d = e_3b = b^2$
We can't solve for b, but we have c, d in terms of b. Thus $c - b = d - b^2 = 0$ guarantees that $\dim I_{(3)} = 3$.

4 Problem 4

The only case we found above gives $f = y^2 + byx + bxy + b^2x^2$ so we get
$g = S_y(f, f) = byxy + bxy^2 + b^2x^2y - by^2x - byxy - b^2yx^2$ and taking a reduction
$h = r_f(g) = bxy^2 + b^2x^2y - b^2yx^2 + b^2yx^2 + b^2xyx + b^3x^3$ and a further reduction
$r_f(h) = b^2x^2y + b^2xyx + b^3x^3 - b^2xyx - b^2x^2y - b^3x^3 = 0$
Thus the reduced Gröbner basis for I consists of just f. If we draw the graph of reduced words and then take the incidence matrix M, one can easily see that:

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

So we can see we are basically in part (i) of problem 1. Thus we get that dimension of the n-th homogenous component of A is the $n + 3$ Fibonacci number, where $A = \mathbb{C}\langle x, y \rangle / I$.