Gröbner Bases - Assignment 4

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1 Problem 1

Let

$$A = F < x, y, z > /(x^2 + yz, x^2 + 3zy)$$

In the last assignment we found that the reduced Gröbner basis G (for our ordering) in question was

$$yz + x^2, \ zy + \frac{x^2}{3}, \ zx^2 - \frac{x^2}{3}, \ yx^2 - 3x^2y$$

So we get that $m = 3$, the maximal degree of a leading monomial. Let V be the set of all reduced monomials of length 2.

$$V = \{x^2, y^2, z^2, xz, xy, zx, yx\}$$

Label those monomials in the order given above, that is x^2 is denoted by 1, xz is denoted by 4. Now for the graph of reduced words Λ, we will use the notation $(i, j) \in G$ means there is a path from i to j.
Now as an example, x^2 and xz have an overlap x and furthermore x^2z is a reduced monomial since it is not divisible by $LM(g)$ for any $g \in G$. Thus $(1, 4) \in \Lambda$.
The full graph of reduced words Λ is:
$x^2 :$ $(1, 1), (1, 4), (1, 5)$
$y^2 :$ $(2, 2), (2, 7)$
$z^2 :$ $(3, 3), (3, 6)$
$xz :$ $(4, 3), (4, 6)$
$xy :$ $(5, 2), (5, 7)$
$zx :$ $(6, 4), (6, 5)$
$yx :$ $(7, 4), (7, 5)$
Thus we can form the incidence matrix M:

$$M = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

We wish to find the characteristic polynomial $p_M(t) = det(t\mathbb{I} - M)$
By Wolfram Alpha we get that

$$p_M(t) = t^7 - 3t^6 + t^5 + 3t^4 - 2t^3$$

Let $n_k(A) = \dim A_k$ where A_k is the space of degree k homogeneous elements in A. Let $N = |V|$ which is, $N = 7$ in this case. By the lectures, we have that

$n_{N+j+m-1}(A) + b_1 n_{N-1+j+m-1}(A) + ... + b_N n_{j+m-1},\ \forall j \geq 0$
where b_i is the co-efficient of the term with degree $N - i$ in the characteristic polynomial
$n_{7+j}(A) - 3n_{6+j}(A) + n_{5+j}(A) + 3n_{4+j}(A) - 2n_{3+j}(A) = 0,\ \forall j \geq 2$

So we now have a recurrence relation for the dimension of the homogeneous components of the quotient algebra. Thus we want to compute M^n for all $n \leq 6, n \in \mathbb{N}$. Then we take the sum of all the entries of M^n to find $n_{n+2}(A)$. That way we can compute $n_{7+j}(A)$ by the recurrence relation. By an online calculator, matrix RESHISH we get:

$$n_2(A) = 7,\ n_{1+2}(A) = 15,\ n_{2+2}(A) = 31,\ n_{3+2}(A) = 63,$$
$$n_{4+2}(A) = 127,\ n_{5+2}(A) = 255,\ n_{6+2}(A) = 511$$

Thus by the recurrence relation we get:

$$n_{7+2} = 3(511) - 255 - 3(127) + 2(63) = 1023$$

And we also get 1023 by the matrix power method. Now we get all the other dimensions of the homogeneous components of A by the recurrence relation.

2 Problem 2

Let (I accidentaly did this question with a z, let's just say it was there all along)

$$A = F < x, y, z > /(x^2y - 2xyx + yx^2 + y, xy^2 - 2yxy + y^2 + x)$$

Let us say we found that the reduced Gröbner basis G (for our ordering) in question was

$$yx^2 - 2xyx + x^2y + y, y^2x - 2yxy + xy^2 + x$$

The maximal degree of a leading monomial is $m = 3$. We get:

$$V = \{x^2, y^2, z^2, xy, xz, yx, yz, zx, zy\}$$

We find that the graph of reduced monomials Λ is:

$$\begin{gathered}(1,1),(1,4),(1,5)\\(2,2),(2,7)\\(3,3),(3,8),(3,9)\\(4,2),(4,6),(4,7)\\(5,3),(5,8),(5,9)\\(6,4),(6,5)\\(7,3),(7,8),(7,9)\\(8,1),(8,4),(8,5)\\(9,2),(9,6),(9,7)\end{gathered}$$

So we get the incidence matrix M:

$$M = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Using WIMS matrix calculator we find that

$$p_A(t) = t^9 - 3t^8 + 2t^6 - t^5$$

So we get the recurrence relation:

$$n_{9+j}(A) - 3n_{8+j}(A) + 2n_{6+j}(A) - n_{5+j}(A) = 0, \ \forall j \geq 2$$

Again using the RESHISH matrix calculator to find $M^n, \ \forall n \leq 8$ we find:

$$\begin{aligned}
&n_2(A) = 9, \ n_3(A) = 25, \ n_4(A) = 70, \ n_5(A) = 195 \\
&n_6(A) = 544, \ n_7(A) = 1517, \ n_8(A) = 4231, \ n_9(A) = 11800 \\
&n_{10}(A) = 32910
\end{aligned}$$

Thus by the recurrence relation we get:

$$n_{11}(A) = 3(32910) - 2(4231) + 1517 = 91785$$

Which is also the answer we get by the matrix power method. We compute the other dimensions of the homogeneous components of A by the recurrence relation.

3 Problem 3

Let us choose the ordering glex $x > y > z$ We will use the Buchberger algorithm to find the reduced Gröbner basis. For our first S-polynomial, formed because of the self overlap of x^2 we get:

$$x(x^2 + yz + zy) - (x^2 + yz + zy)x = xyz + xzy - yzx - zyx$$
$$xyz + xzy - yzx - zyx - (xy + yx + z^2)z = xzy - yzx - zyx - yxz - z^3$$
$$xzy - yzx - zyx - yxz - z^3 - (xz + y^2 + zx)y = -yzx - zyx - yxz - z^3 - y^3 - zxy$$
$$-yxz - yzx - zyx - z^3 - y^3 - zxy + y(xz + y^2 + zx) = -zyx - z^3 - zxy$$
$$-zxy - zyx - z^3 + z(xy + yx + z^2) = 0$$

Similarly the S-polynomials that are formed because of the x overlap of x^2, xz and x^2, xy go to 0. Thus the reduced Gröbner basis G is:

$$x^2 + yz + zy, xy + yx + z^2, xz + y^2 + zx$$

Thus the maximum degree of a leading monomial is $m = 2$ and we also get

$$V = \{x, y, z\}$$

So we get the graph of reduced words Λ:

$$(2,1), (2,2), (2,3)$$
$$(3,1), (3,2), (3,3)$$

Thus the incidence matrix M is:

$$M = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

So we can easily see that $p_A(t) = t((t-1)^2 - 1) = t^3 - 2t^2$. Thus:

$$n_{3+j}(A) - 2n_{2+j}(A) = 0, \ \forall j \geq 1$$

Using the incidence matrix we can compute by hand:

$$n_1(A) = 3, \ n_2(A) = 6, \ n_3(A) = 6(2) = 12$$

Thus by the recurrence relation $n_4(A) = 24$. We get the other dimensions of the homogenous components of A by the recurrence relation.

$$n_i(A) = 2^{i-1}(3), \forall i \geq 1$$

4 Problem 4

For $I \subset T(X)$, $G \subset I$ is a Gröbner basis iff the cosets of monomials that are reduced with respect to G form a basis of the quotient $T(X)/I$
For $I = (xyz)$ we get the Gröbner basis G consisting of xyz.
The reduced monomials of degree n, $\forall n \geq 1$ with respect to G are of the form

$$x^i y^j z^k \text{ such that } ijk \neq 0, i + j + k = n$$

So it is the cosets of these monomials that form a basis of the quotient $F[x, y, z]/I$. Also by the above formulation, we can see that $n_k(A) = 3k$, $\forall k \geq 1$. Now

$$h_A(t) = 1 + \sum_{k \geq 1} 3kt^k$$

This Hilbert series is a rational function. Now, I will admit that for the second part, the way I want to do it is to compute the hilbert series and show it is not a rational function. Thus there does not exist an admissible order of monomials for which the non-commutative algebra has a finite Gröbner basis. However I am not sure how to compute the Hilbert series in this case. By investigating the truncated Gr'öbner basis I was able to find the reduced monomials for some degrees, but I couldn't quite figure out the pattern to generalise it.