Gröbner Bases - Assignment 3

Sean Martin

25/02/2016

1 Problem 1

In the last assignment we made the conjecture that the reduced Gröbner basis G would consist of $k_0 = yxy - xyx$ and polynomials of the form
$k_n = yx^nyx - xyx^2y^{n-1},\ n > 1, n \in \mathbb{N}$.
Firstly we must show that our proposed basis G generates the same ideal as k_0. To do this I will show that we can obtain $k_n, n > 1$ using k_0 using sums and products:

$$(-yx^{n-1})(yxy - xyx) = yx^nyx - yx^{n-1}yxy = g_1$$
$$g_1 - yx^{n-2}(yxy - xyx)y = yx^nyx - yx^{n-2}yxy^2 = g_2$$
$$g_2 - yx^{n-3}(yxy - xyx)y^2 = yx^nyx - yx^{n-3}yxy^3 = g_4$$
and we continue similarly to obtain $yx^nyx - yxyxy^{n-1} = g_{n-2}$
$$g_{n-2} + (yxy - xyx)xy^{n-1} = yx^nyx - xyx^2y^{n-1} = k_n \text{ as required}$$

Now we must check if we get 0 on reduction by G for all possible S-polynomials. The possible overlaps we can get between these polynomials are:
1) The self overlap y of k_0
2) The overlap y between k_0 and $k_n, \forall n > 2$
3) The overlap yx between k_n and $k_p, \forall n \neq 0, p \neq 0$
4) The overlap yx between k_n and $k_0, \forall n \neq 0$
We get the following S-polynomials:
1) $LM(k_0) = yxy$ so we get:
$S_y(k_0, k_0) = k_0(xy) - (yx)k_0 = -xyx^2y + yx^2yx = k_2$ so we get a result of 0 on reduction by G as required for the Diamond Lemma.

2) $LM(k_0) = yxy,\ LM(k_n) = yx^nyx$ so we get:
$S_y(k_0, k_2) = k_0(x^nyx) - yx(k_2) = -xyx^{n+1}yx + yx^2yx^2y^{n-1} = g_1$ which is divisible by k_2. On reduction we get
$r_{k_2}(g_1) = g_1 - k_2(xy^{n-1}) = -xyx^{n+1}yx + xyx^2yxy^{n-1} = g_2$ which is divisible by k_2. On reduction we get
$r_{k_2}(g_2) = g_2 - x(k_2)y^{n-1} = -xyx^{n+1}yx + x^2yx^2y^n = -g_3$ and g_3 is divisble by k_{n+1}. On reduction we get
$r_{k_{n+1}}(g_3) = g_3 - xk_{n+1} = -x^2yxy^n + x^2yx^2y^n = 0$ as required for the Diamond Lemma.

3) $LM(k_n) = yx(x^{n-1}yx)$, $LM(k_p) = yx^p(yx)$, so k_n, k_p have overlap yx. We get:
$S_{yx}(k_p, k_n) = k_p(x^{n-1}yx) - (yx^p)k_n = -xyx^2y^{p-1}x^{n-1}yx + yx^{p+1}yx^2y^{n-1} = g_1$ which is divisible by k_{p+1}. On reduction we get
$r_{k_{p+1}}(g_1) = g_1 - (k_{p+1})(xy^{n-1}) = -xyx^2y^{p-1}x^{n-1}yx + xyx^2y^pxy^{n-1} = g_2$ which is divisible by k_0. On reduction we get
$r_{k_0}(g_2) = g_2 - xyx^2y^{p-1}(k_0)(y^{n-2}) = -xyx^2y^{p-1}x^{n-1}yx + xyx^2y^{p-1}xyxy^{n-2} = g_3$. $LM(g_3) = xyx^2y^{p-1}xyxy^{n-2}$ and so we can perform another reduction. We obtain:
$r_{k_0}(g_3) = -xyx^2y^{p-1}x^{n-1}yx + xyx^2y^{p-1}x^2yxy^{n-3} = g_4$ which again is divisible by k_0. We can reduce g_4 by k_0 another $n-3$ times to finally obtain
$-xyx^2y^{p-1}x^{n-1}yx + xyx^2y^{p-1}x^{n-1}yx = 0$ as required for the Diamond Lemma.

4) $LM(k_0) = (yx)y$, $LM(k_n) = yx^n(yx)$ so we get:
$S_{yx}(k_n, k_0) = k_n(y) - (yx^n)k_0 = -xyx^2y^n + yx^{n+1}yx = k_{n+1}$ so we get a result of 0 on reduction by G as required for Diamond Lemma.

2 Problem 2

So from last week, we had that the elements of the Gröbner basis of degree up to 5 were:

$$xz^2yx - z^2yz, yzyx - xy^2z, y^3z, xzyx - zyz, y^2x, xyx - yz, zx - xz$$

So we want to explain the reduced monomials of degree up to 5. They can't contain the words:

$$zx,\ xyx,\ y^2x,\ xzyx,\ y^3z,\ yzyx,\ xz^2yx$$

We work out how many reduced monomials there are of each degree by subtracting the number combinations which give non-reduced monomials from the total number of possible monomials of said degree and adding back on any monomials we double counted:
Of degree 1, there is 3, x, y, z
Of degree 2, there is $3^2 - 1 = 8$
Of degree 3, there is $3^3 - 2 - (3)(2) = 19$
Of degree 4, there is $3^4 - 3 - (3)(4) - (3)(3^2) + 1 = 40$ where we add 1 becuase otherwise we would be double subtracting the case of the non-reduced monomial $zxzx$.
Of degree 5, (the idea of this is right, the figures are not and I don't feel like fixing the figures right now)there is $3^5 - 1 - 3(2)(3) - 3(3^2) + 1 - 4(3^3) + y$ where y is determined by the number of times we subtract a duplicate for zx. $zxabc$ gives you 2 non-reduced monomials containing the word zx twice. Namely $ab = zx$, and $bc = zx$ with ab. $azxbc$ gives you 1. $abzxc$ gives you one monomial you already considered. $abczx$ gives you two monomials you already considered. So y = 3. Thus there is 93 reduced monomials of degree 5.
Overall we get 171 reduced monomials out of a possible $363 = 3 + 3^2 + 3^3 + 3^4 + 3^5$.

3 Problem 3

We want to compute the reduced Gröbner Basis for the ideal

$$(x^2 + yz, x^2 + 3zy) \subset F < x, y, z >$$

Let us choose the ordering glex, $x < y < z$ so that we don't have to consider the self overlaps of x^2.
Converting the polynomials that generate our ideal to standard form we get

$$g_1 = zy + \frac{x^2}{3},\ g_2 = yz + x^2$$

We have $LM(g_1) = zy, LM(g_2) = yz$. These have an overlap y and an overlap z. Lets first consider the y overlap and the corresponding S-polynomial:

$$S_y(g_2, g_1) = g_1(z) - z(g_2) = \frac{x^2 z}{3} - zx^2 = f_1$$

f_1 is reduced with respect to g_1, g_2, and so we get a new polynomial in our Gröbner basis, $-f_1 = zx^2 - \frac{x^2 z}{3}$. Now we take the other S-polynomial.

$$S_z(g_1, g_2) = g_2(y) - y(g_1) = x^2 y - \frac{yx^2}{3} = f_2$$

f_2 is reduced with respect to g_1, g_2 and so we get a new polynomial in our Gröbner basis, $yx^2 - 3x^2 y$.
We don't get any more overlaps except for $LM(g_2), LM(f_1)$ which have an overlap z.

$$S_z(g_2, -f_1) = g_2(x^2) - y(-f_1) = x^4 + \frac{yx^2 z}{3} = p_1$$

Now we perform long division: $LM(p_1) = yx^2 z$ is divisible by $LM(f_2) = yx^2$, taking a reduction

$$r_{f_2}(p_1) = p_1 + f_2(z) = x^4 + x^2 yz = p_2$$

Now $LM(p_2) = x^2 yz$ is divisible by $LM(g_2) = yz$, taking a reduction

$$r_{g_2}(p_2) = p_2 - (x^2) g_2 = x^4 - x^4 = 0$$

Since we get 0 on long division we don't get another term in our Gröbner basis. Thus our reduced Gröbner basis is:

$$yz + x^2,\ zy + \frac{x^2}{3},\ zx^2 - \frac{x^2 z}{3},\ yx^2 - 3x^2 y$$

The reduced monomials with respect to this basis are:

$$x^{n_1} z^{n_{21}} y^{n_{22}} x z^{n_{31}} y^{n_{32}} x ... z^{n_{k1}} y^{n_{k2}} x^l$$

With n_{i1} or $n_{i2} = 0, \ \forall i,$ but not both zero $n_1, n_{i1}, n_{i2} \in \mathbb{N},$ (with $0 \in \mathbb{N}$) and $l = 0$ or 1.

4 Problem 4

We will choose the glex order, $x < y$. We want to compute the Gröbner basis for the ideal generated by

$$g_1 = yx^2 - 2xyx + x^2y + y,\ \ g_2 = y^2x - 2yxy + xy^2 + x$$

So $LM(g_1) = yx^2,\ LM(g_2) = y^2x$. The only overlap we get of these is yx. $LM(g_1) = (yx)x,\ LM(g_2) = y(yx)$. So we get the S-polynomial:

$$\begin{aligned} S_{yx}(g_2, g_1) = g_2(x) - y(g_1) &= -2yxyx + xy^2x + x^2 + 2yxyx - yx^2y - y^2 \\ &= -yx^2y + xy^2x + y^2 - x^2 = -f_1 \end{aligned}$$

So $LM(f_1) = yx^2y$ which is divisible by $LM(g_1) = yx^2$ so we take a reduction:

$$\begin{aligned} r_{g_1}(f_1) = f_1 - g_1(y) &= -xy^2x + y^2 - x^2 - x^2y^2 + 2xyxy - y^2 \\ &= -xy^2x - x^2y^2 + 2xyxy - x^2 = -f_2 \end{aligned}$$

So $LM(f_2) = xy^2x$ which is divisible by $LM(g_2) = y^2x$ so we take a reduction:

$$r_{g_2}(f_2) = f_2 - x(g_2) = x^2y^2 - 2xyxy + x^2 + 2xyxy - x^2y^2 - x^2 = 0$$

So we get a result of 0 on long division, so we don't get any new elements in the Gröbner basis. Thus our reduced Gröbner basis is:

$$yx^2 - 2xyx + x^2y + y, y^2x - 2yxy + xy^2 + x$$

The reduced monomials are:

$$x^n(yx)^q y^p, \forall n, p, q \in \mathbb{N} \text{ (with } 0 \in \mathbb{N})$$

.

5 Problem 5

5.1 Part a

This follows from part (b) becuase there is infinitely many ways to have the element a_n defined for all n for the recurrence relation $a_{n+1} = \frac{1+a_n}{a_{n-1}}$.

5.2 Part b

We have the recurrence relation

$$a_{n+1} = \frac{1+a_n}{a_{n-1}} \text{ for some given } a_0, a_1$$

We will show that $a_5 = a_0$ and that $a_6 = a_1$ and thus the relation repeats and $a_{k+5} = a_k,\ \forall k$. Consider a_3, a_4, a_5, a_6 where none of these are zero (and $a_0, a_1 \neq 0, a_1 \neq -1$):

$$\begin{aligned}
a_3 &= \frac{1+a_2}{a_1} = \frac{1+\frac{1+a_1}{a_0}}{a_1} \\
&= \frac{a_0+a_1+1}{a_0a_1} \\
a_4 &= \frac{1+a_3}{a_2} = \frac{1+\frac{a_0+a_1+1}{a_0a_1}}{\frac{1+a_1}{a_0}} \\
&= \frac{a_0a_1+a_0+a_1+1}{a_1^2+a_1} \\
a_5 &= \frac{1+a_4}{a_3} = \frac{1+\frac{a_0a_1+a_0+a_1+1}{a_1^2+a_1}}{\frac{a_0+a_1+1}{a_0a_1}} \\
&= \frac{a_0(a_1^3+2a_1^2+a_1+a_0a_1^2+a_0a_1)}{a_1^3+2a_1^2+a_1+a_0a_1^2+a_0a_1} \\
&= a_0 \\
a_6 &= \frac{1+a_5}{a_4} = \frac{1+a_0}{a_4} \\
&= (1+a_0)\frac{a_1^2+a_1}{a_0a_1+a_0+a_1+1} \\
&= \frac{a_1(a_1+a_0a_1+a_0+1)}{a_1+a_0a_1+a_0+1} \\
&= a_1
\end{aligned}$$

Since $a_5 = a_0, a_6 = a_1$ we get that $a_7 = a_2, a_8 = a_3, ...$ as was required.

The relation to the computation in the first part of the question is as follows. Assume that $x, y, z, t, u \neq 0$, then they act like $a_0, a_1, a_2, a_3, a_4, a_5$ respectively.

$$\begin{aligned}
xz = y + 1 &\implies z = \frac{y+1}{x} = a_2 \\
yt = z + 1 &\implies t = \frac{z+1}{y} = a_3 \\
zu = t + 1 &\implies u = \frac{t+1}{z} = a_4 \\
tx = u + 1 &\implies x = \frac{u+1}{t} = a_5 = a_0 \\
uy = x + 1 &\implies y = \frac{x+1}{u} = a_6 = a_1
\end{aligned}$$