Gröbner Bases - Assignment 2

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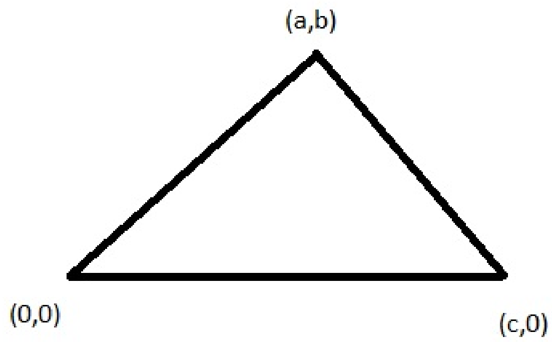
1 Problem 1

Without loss of generality, we can present any triangle as the following:



Finding the equations of the altitudes will rely on the following: if line A is perpendicular to line B and line A has slope $m_1 = \frac{a}{b}$ then line B has slope $m_2 = \frac{-b}{a}$.
Then we have the equations of the three altitudes of the triangle: For $(0, 0)$ the slope of the line its corresponding altitude is perpendicular to has slope $\frac{-b}{c-a}$ so we get slope $\frac{c-a}{b}$ for the altitude. Thus its equation is: $y = \frac{c-a}{b}x$.
For (a, b) we get the equation $x = a$ since the slope of the line is undefined.
For $(b, 0)$ we get the eqaution $y = \frac{-a}{b}(x - c)$
Let $\frac{1}{b}$ be denoted by b for simplicity. Now we use SINGULAR to get a Gröbner basis for the ideal generated by the commutative polynomials
$x - a, y + (a - c)bx, y + ab(x - c)$.
Unfortunately not notation in SINGULAR is a bit messed up. The c in SINGULAR is the b above, and the b in SINGULAR is the c above.

```
>                               SINGULAR
 A Computer Algebra System for Polynomial Computations

 by: W. Decker, G.-M. Greuel, G. Pfister, H. Schoenemann
FB Mathematik der Universitaet, D-67653 Kaiserslautern
> ring r = (0,a,b,c), (x,y), dp;
> ideal I = x- a,
. y + a * c * (x - b),
. y + (a-b) * c * x;
> option(redSB);
> ideal G = groebner(I);
> G;
G[1]=y+(a2c-abc)
G[2]=x+(-a)
```

Since we can find a common solution to the two polynomials in the Gröbner basis when $b \neq 0$ we can say the three altitudes in a generic triangle are concurrent.

2 Problem 2

In this case we use Magma to compute the reduced Gröbner basis for degrees up to 10 for the ideal given in the question. We obtain the following:

```
Q := RationalField();
P<y,x> := FreeAlgebra(Q,2,"glex");
B := [x*y*x - y*x*y];
for i :=2 to 10 do
I := GroebnerBasis(B,i);
i;
I;
end for;
```

Clear

```
4
[
    y*x*y - x*y*x
]
5
[
    y*x^2*y*x - x*y*x^2*y,
    y*x*y - x*y*x
]
6
[
    y*x^3*y*x - x*y*x^2*y^2,
    y*x^2*y*x - x*y*x^2*y,
```

And we can see a developing pattern, even in the image. At degree 5 we obtain a new term of the form $yx^2yx - xyx^2y$ and then at degree 6 we get a new term of the form $yx^3yx - xyx^2y^2$.
Investigating higher degrees, we make the conjecture that the full Gröbner basis will consist of $yxy - xyx$ and polynomials $yx^nyx - xyx^2y^{n-1}$ for all $n \in \mathbb{N}, n \geq 2$.

3 Problem 3

I used the following code in Magma to compute the reduced grobner basis for the given ideal for degrees up to ten. The result was very long and quite complex, I haven't fully figured out the full Gröbner basis. I did figure out the following though by looking at the reduced Gröbner basis for different degrees.
The full Gröbner basis will consist of the following polynomials: (let $n, p \in \mathbb{Z}$)
$zx - xz$,
y^2x
$xz^{n-3}yx - z^{n-3}yz$ for $n \geq 3$,
$yz^{\frac{n-5}{2}}yz^{p-n}yz^{\frac{n-3}{2}}$ for $n \geq 5$, n odd, $p \geq n$
$yz^{n-4}yx - xyz^{n-5}yz$ for $n \geq 5$,
$yxz^{\frac{n-6}{2}}yz^{p-n}yz^{\frac{n-2}{2}}$ for $n \geq 6$, n even, $p \geq n$
$yx^2z^{\frac{n-8}{2}}yz^{p-n}yz^{\frac{n-2}{2}}$ for $n \geq 8$, n even, $p \geq n$
$yx^{\frac{n-4}{2}}yz^{p-n}yz^{\frac{n-2}{2}}$ for $n \geq 6$ n even, $p \geq n$

4 Problem 4

We use Magma to compute the reduced Gröbner basis and the dimension of the quotient.

```
Q := RationalField();
P<y,x> := FreeAlgebra(Q,2,"glex");
I := ideal<P|x^2 -1, y^5 - 1, x*y - y^4*x>;
B := GroebnerBasis(I);
B;
B := quo<P|I>;
Dimension(B);
```

Clear

```
[
    y^4 - x*y*x,
    y^3*x - x*y^2,
    x*y^3 - y^2*x,
    x*y^2*x - y^3,
    y*x*y - x,
    x^2 - 1
]
10
```

5 Problem 5

5.1 Part a

We can make readily see that we can make the association that vertice i in the graph Λ can be one of k possible colours iff $x_i^k = 1$ because $x_i^k = 1$ has k distinct roots. And we can associate each colour to a complex root of $x_i^k - 1$ Now let us say that two vertices i and j are connected by an edge in Λ and are of k possible colours. That is iff (Statement A)

$$x_i^k - x_j^k = (x_i - x_j)(x_i^{k-1} + x_i^{k-2}x_j + ... + x_i x_j^{k-2} + x_j^{k-1}) = 0$$

Where the above polynomials are communtative. If we require the vertices i, j to have different colours then we require that $x_i \neq x_j$. So in this case A is true iff $x_i^{k-1} + x_i^{k-2}x_j + ... + x_i x_j^{k-2} + x_j^{k-1}) = 0$. Since a regular colouring would follow the process above for each vertice i and pair of vertices i, j we obtain the result. The polynomials in the collection in the question, C_Λ have a common zero iff Λ admits a regular colouring of vertices in k colours.

5.2 Part b

We used Magma to compute the reduced Grobner basis for the commutative polynomials in C_Λ in the graph given in the question with 8 vertices.

```
Q := RationalField();
P<x1,x2,x3,x4,x5,x6,x7,x8> := PolynomialRing(Q,8);
B := [x1^3 - 1, x2^3 - 1, x3^3-1, x4^3 -1, x5^3 -1, x6^3 -1, x7^3 -1, x8^3
x3 + x3^2, x1^2 + x1*x4 + x4^2, x1^2 + x1*x5 + x5^2,x2^2 + x2 * x8 + x8^2,
+ x4^2, x2^2 + x2 * x7 +x7^2, x3^2 + x3 * x4 + x4^2, x3^2 +x3 * x8 + x8^2,
+ x8^2, x4^2 + x4 * x5 + x5^2, x5^2 + x5 * x6 + x6^2, x6^2 +x6*x7 + x7^2,
8^2, x7^2 + x7*x8 + x8^2] ;
I := GroebnerBasis(B);
I;
```

Clear

```
[
    x1 + x7 + x8,
    x2 + x7 + x8,
    x3 - x7,
    x4 - x8,
    x5 - x7,
    x6 + x7 + x8,
    x7^2 + x7*x8 + x8^2,
    x8^3 - 1
]
```

The results show that there does exist a regular colouring of vertices of the graph in question since we can find a common zero of the polynomials in the

Gröbner basis. I will the desribe the possible colourings:
$x_8^3 - 1 = 0 \implies$ vertex 8 can be any of the three possible colours.
$x_4 - x_8 = 0 \implies$ vertex 4 must be the same colour as 8.
$x_3 - x_7 = x_5 - x_7 = 0 \implies$ vertices 3,5,7 must have the same colour. This colour must not be the same as 8 because $x_7^2 + x_7 * x_8 + x_8^2 = 0$.
Finally $x_1 + x_7 + x_8 = x_2 + x_7 + x_8 = x_6 + x_7 + x_8 = 0 \implies$ vertices 1, 2, 6 must have the same colour and a different colour to 8 and 7.
This is the only possible situation to get a regular colouring. Thus say our three colours are R,G,B, then up to permuation the regular colouring is (where = denotes of that colour):

$$1 = G, 2 = G, 3 = R, 4 = B, 5 = R, 6 = G, 7 = R, 8 = B$$