Gröbner Bases - Assignment 1

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1 Problem 1

1.1 Part a

glex: compare degree, then compare term by term. So for the given polynomial:

$$2x_1x_2 + 3x_2x_1 + x_1x_3 + x_2^3 - x_1^2x_3^2 + x_3^3$$

In standard glex order:

$$-x_1^2x_3^2 + x_3^3 + x_2^3 + 3x_2x_1 + x_1x_3 + 2x_1x_2$$

1.2 Part b

In this case we get:

$$-x_1^2x_3^2 + x_2^3 + x_3^3 + 2x_1x_2 + x_1x_3 + 2x_2x_1$$

1.3 Part c

We have that $wt(x_1) = 4, wt(x_2) = 2, wt(x_3) = 1$, so:
$wt(x_1x_2) = 4 + 2 = 6 = wt(x_2x_1)$
$wt(x_1x_3) = 4 + 1 = 5$
$wt(x_2^3) = 2 + 2 + 2 = 6$
$wt(x_1^2x_3^2) = 2(4) + 2(1) = 10$
$wt(x_3^3) = 3(1) = 3$

Then we get the order $glex_{(4,2,1)}$ by comparing weights, and then terms (with $x_1 < x_2 < x_3$):

$$-x_1^2x_3^2 + x_2^3 + 3x_2x_1 + 2x_1x_2 + x_1x_3 + x_3^3$$

2 Problem 2

2.1 Part a

First let us show that $\prec$ is a total order:
Assume that $m \neq m'$ are two non-commutative monomials in x_1, x_2. Then there are two possible cases:
1) m has a different number of occurences of x_2 to m' so either $m \prec m'$ or $m' \prec m$.
2) They have the same number of x_2, in which we have two cases (Since $m \neq m'$):
2A) Without loss of generality, we can say that m is m' appended with any positive number of x_1 occurences. That is $m = m'm'', m'' \neq 1$. So $m' \prec m$
2B) m and m' differ at some position. Take the first such position. Without loss of generality, m must have an x_1 and m' must have an x_2 in this position, since they differ. So $m' \prec m$.
This shows that $m \neq m' \implies m \prec m'$ or $m' \prec m$ that is, we have a total order.

$\prec$ is a well order: (argument *)
Take any set S of non-commutative monomials in x_1, x_2. Then there are two possible cases:
1) There is a unique element of S with the minimal number of x_2 occurences, and thus a unique minimal element of S.
2) There is no unique element of S with the minimal number of x_2 occurences, in which there are two cases: (Call $S' \subset S$ the subset of elements with the minimal number of x_2 occurences)
2A) There is one element $s \in S'$ such that $\forall x \in S', x = sx', x' \neq 1$. That is, we can append x_1 occurences to s to obtain all other elements of S'. So there is a unique minimal element of S' and so a unique minimal element of S.
2B) There is no element s as described above. So there must be an element $s \in S'$ which differs to at least one other element $s' \in S'$ at some position. Take the first position where we find at least two differing elements of S' and call this position i. Reduce S' to S'' where $S'' \subset S' \subset S$ is the subset of S' consisting of monomials with the letter x_2 in position i. Then $x \prec y, \forall x \in S'',\ y \in S' \setminus S''$.

We can give the same argument * for the set S'' instead of S to get by 2A) a unique minimal element of S'' or by 2B) $S''' \subset S''$ which is further reduced. If we get a further reduced set, then give argument * again to get

a further reduced set, and so forth.
All that remains to be said is that we will eventually get a unique minimal element in the reduced sets obtained from 2B). I can see this to be true because 2B) would eventually have to reduce to a singleton, or a set with one element s that can produce all other elements of the set by appending x_1 occurences to s. However I am having trouble presenting this as any kind of formal argument. Finally we obtain that S has a unique minimal element, and since S was arbitrary we have a well order.

$\prec$ is admissible:
Let m_1, m_2, m_3 be monomials in x_1, x_2. Let $m_1 \prec m_2$, which gives three possibilities:
1) If m_1 has fewer occurences of x_2 than m_2. Then m_3m_1 has fewer occurences of x_2 than m_3m_2 and m_1m_3 has fewer occurences of x_2 than m_2m_3. So order is maintained under multiplication.
2) If m_1 has the same number of occurences of x_2 as m_2, and $m_2 = m_1m'$ for some $m' \neq 1$. Then $m_3m_2 = m_3m_1m'$ so order is preserved by left multiplication. For right multiplication, if $m_3 = x_1^n$ then order is easily seen to be preserved - we are just appending more x_1 occurences. Otherwise, m_3 has at least one occurence of x_2. In this case in the first position m_1m_3 and $m_2m_3 = m_1m'm_3$ differ, m_1m_3 will have the letter x_2. This is becuase m' consists of at least one x_1 occurence so m_3 must have an x_2 term before $m'm_3$. Thus order is preserved.
3) If m_1 has the same number of occurences of x_2 as m_2, and in the first position they differ, m_1 has the letter x_2. Then multiplication can't change this. Since we are multiplying m_1 and m_2 by m_3 and m_3 won't have any different letters to itself.

2.2 Part b

There are $3! = 6$ permutations of x_1, x_2, x_3 so 6 possible orderings, which we will check to see if an admissable order exists. Assume we have an admissable order such that $x_1x_2 > x_3^2$, $x_2x_3 > x_1^2$, $x_3x_1 > x_2^2$. We will extensively use the fact that an admissable order maintains order under multiplication.
Assume that $x_1 > x_2 > x_3$ so $x_1^2 > x_3x_1 > x_2^2 > x_2x_3 > x_1^2$ so $x1^2 > x_1^2$ which is a contradiction.
Assume that $x_1 > x_3 > x_2$ so $x_1^2 > x_1x_2 > x_3^2 > x_2x_3 > x_1^2$ contradiction.
Assume that $x_2 > x_1 > x_3$ so $x_2^2 > x_2x_3 > x_1^2 > x_3x_1 > x_2^2$ contradiction.
Assume that $x_2 > x_3 > x_1$ so $x_2^2 > x_1x_2 > x_3^2 > x_3x_1 > x_2^2$ contradiction.
Assume that $x_3 > x_1 > x_2$ so $x_3^2 > x_2x_3 > x_1^2 > x_1x_2 > x_3^2$ contradiction.
Assume that $x_3 > x_2 > x_1$ so $x_3^2 > x_3x_1 > x_2^2 > x_1x_2 > x_3^2$ contradiction.

So we get a contradiction for every possible ordering of x_1, x_2, x_3. Thus our original assumption that we had an admissable order such that
$x_1x_2 > x_3^2,\ x_2x_3 > x_1^2,\ x_3x_1 > x_2^2$ was false. Hence no such admissable order exists.

3 Question 3

3.1 Part a

Firstly we will consider standard glex order, so that $LM(x_1x_2 - x_2x_1) = x_2x_1$.
Long division algorithm, dividing $f = x_2x_1x_2x_1$ by $g = x_1x_2 - x_2x_1$:
$LM(f) = f$ is divisible by $LM(g), f = m'LM(g)m''$ so replace f by $r_g(f) = f - \frac{LC(f)}{LC(g)}m'gm''$ as follows:

$$f = LM(g)x_2x_1 \implies r_g(f) = f - \frac{1}{-1}gx_2x_1 = f - f + x_1x_2x_2x_1$$

Let $f = r_g(f)$ (the programming type of equal) from the last step. Again $LM(f) = f$ is divisible by $LM(g)$ so replace f by $r_g(f)$ as follows:

$$f = x_1x_2LM(g) \implies r_g(f) = f - \frac{1}{-1}x_1x_2g = f - f + x_1x_2x_1x_2$$

Let $f = r_g(f)$ from the last step. Again $LM(f) = f$ is divisible by $LM(g)$ so replace f by $r_g(f)$:

$$f = x_1LM(g)x_2 \implies r_g(f) = f - \frac{1}{-1}x_1gx_2 = f - f + x_1x_1x_2x_2$$

At this point $f = r_g(f)$ is not divisible by LM(g), and $f - LT(f) = 0$, so return $LT(f) = f = x_1x_1x_2x_2$

We can do this long division again, with $f = x_2x_1x_2x_1$ divided by $g = x_1x_2 - x_2x_1$:

$$LM(f) = f = x_2x_1LM(g) \implies r_g(f) = f + x_2x_1g = x_2x_1x_1x_2$$

Let $f = r_g(f)$ from the last step.

$$LM(f) = f = LM(g)x_1x_2 \implies r_g(f) = f + gx_1x_2 = x_1x_2x_1x_2$$

As before, we do one more reduction (the same one as before) to return the same answer: $x_1x_1x_2x_2$

Now we could change the ordering so that $LM(g) = x_1x_2$ and then division of f by g would proceed as follows:

$$LM(f) = f = x_2LM(g)x_1 \implies r_g(f) = f - \frac{1}{1}x_2gx_1 = x_2x_2x_1x_1$$

So we return the answer $x_2x_2x_1x_1$, which is different. So it would seem that the long division algorithm only produces a unique answer for a fixed ordering.

3.2 Part b

Let $f = x_2^3$, $g = x_1^2 - x_1x_2 + x_2^2$ and consider standard glex order, so $LM(g) = x_2^2$. So long division proceeds as follows, dividing f by g:

$$LM(f) = f = x_2LM(g) \implies r_g(f) = f - \frac{1}{1}x_2g = -x_2x_1^2 + x_2x_1x_2$$

Now let $f = r_g(f)$ from the last step. $LM(f)$ is not divisible by $LM(g)$. And $f - LT(f) = -x_2x_1^2$ is also not divisible by $LM(g)$ so we return $LT(f) = x_2x_1x_2$.

We could also do long division as follows for standard glex:

$$LM(f) = f = LM(g)x_2 \implies r_g(f) = f - gx_2 = -x_1^2x_2 + x_1x_2^2$$

Let $f = r_g(f)$ from the last step.

$$LM(f) = x_1x_2^2 = x_1LM(g) \implies r_g(f) = f - x_1g = -x_1^2x_2 - x_1^3 + x_1^2x_2 = -x_1^3$$

Which is different, so it would seem that long division does not produce a unique answer.

x_2^3 is not divisible by x_1^2 or x_1x_2 so any other ordering with $LM(g) \neq x_2^2$ will just return x_2^3 at the end.

4 Question 4

4.1 Part a

Let $I = (x_1 - x_2, x_1 - x_3)$ and $F' = F\langle x_1, x_2, x_3\rangle$. Then
$x_1 - x_2 \in I \implies 0 + I = x_1 - x_2 + I = x_1 + I - (x_2 + I) \implies x_1 + I = x_2 + I$,

that is, x_1, x_2 are equivalent. Similarly $x_1 - x_2 \in I \implies x_1 + I = x_3 + I$. So we have that $x_1 + I = x_2 + I = x_3 + I$, thus cosets of powers of x_1 form a spanning set in the quotient.
Thus we can consider x_1 to be just x and form a map $f : F' \mapsto F[x], x_i \to x$ which is well defined and surjective homomorphism. Since f is clearly surjective, $F'/kerf \simeq F[x]$ by the first isomorphism theorem. Thus $F'/I \simeq F[x]$ as required since $I \subset kerf$ and this shows linear independence of the cosets of powers of x_1.

4.2 Part b

We will use the result shown in class, that $G \subset I$ is a Gröbner Basis of I iff cosets of monmials reduced with respect to G form a basis of $F\langle x_1, ..., x_n\rangle/I$. Let $G = \{x_1 - x_2, x_1 - x_3\}$ and I be the ideal G generates, so $G \subset I$. Firstly consider standard glex order:
In this case, $LM(x_1 - x_2) = x_2$, $LM(x_1 - x_3) = x_3$ so the monomials of the form x_1^n are reduced with respect to G. The cosets of these monomials, $x_1^n + I$ span $F[x]$ and are linearly independent, so form a basis of $F[x] \simeq F\langle x_1, x_2, x_3\rangle/I$ by part a. Thus G is a Gröbner Basis of I.

Consider now reverse glex order:
In this case, $LM(x_1 - x_2) = x_1, LM(x_1 - x_3) = x_1$ so the monomials of $F\langle x_1, x_2, x_3\rangle$ which do not contain an x_1 term are reduced with respect to G. The cosets of these monomials will not be linearly independent since $a, b \in F$, $a(x_2 + I) + b(x_3 + I) = 0$ for $a = -b$ since $x_2 - x_3 \in I$ and x_2, x_3 are reduced monomials with respect to G. Thus these monomials can't form a basis. Thus G is not a Gröbner Basis of I for reverse glex order.