Calculus for Engineers II - Sample Problems on Integrals

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Question 1: Solve the following integrals:

1. $$\int_0^\pi \sin^2 x dx$$

2. $$\int \frac{dx}{x^2-4}$$

3. $$\int \sinh^8 x \cosh x dx$$

4. $$\int \sin x \cos^7 x dx$$

5. $$\int_1^2 dx x^5 \ln x$$

6. $$\int_0^1 \frac{8x+6}{\sqrt{3x+2x^2}} dx$$

7. $$\int x e^{-x} dx$$

8. $$\int \frac{9y dy}{2y^2+3}$$

9. $$\int \frac{dx}{x^4-1}$$

10. $$\int \frac{\ln^3 x}{x} dx$$

11.

$$\int \frac{x^2}{(x-1)(x-2)}$$

12.

$$\int \frac{10dx}{x^2+9}$$

Solutions

Question 1.1

$$\int_0^{\pi} \sin^2 x dx$$

Solution:
The easiest way to solve this integral is by using the trigonometric identity $\sin^2 x = \frac{1-\cos 2x}{2}$.

$$\int_0^{\pi} \sin^2 x dx = \int_0^{\pi} \frac{1-\cos 2x}{2} dx = \int_0^{\pi} \frac{1}{2} dx - \int_0^{\pi} \frac{\cos 2x}{2} dx$$

We have now two integrals to solve; the first one is trivial

$$\int_0^{\pi} \frac{1}{2} dx = \frac{1}{2} x|_0^{\pi} = \frac{\pi}{2}$$

whereas the second one is:

$$\int_0^{\pi} \frac{\cos 2x}{2} dx$$

To solve the second integral we change variable from x to u using $u = 2x \Rightarrow dx = \frac{du}{2}$. We should accordingly change the limits of integration, from $(0, \pi)$ to $(0, 2\pi)$ for the new variable u.

$$\int_0^{\pi} \frac{\cos 2x}{2} dx = \frac{1}{4} \int_0^{2\pi} \cos u du = \frac{1}{4} (\sin x)_0^{2\pi} = \frac{1}{4}(0-0) = 0$$

Combining the two integrals we obtain

$$\int_0^{\pi} \sin^2 x dx = \frac{\pi}{2}$$

Question 1.2

$$\int \frac{dx}{x^2-4}$$

Solution:
This integral is solved by decomposing the function into partial fractions, i.e.,

$$\frac{1}{x^2-4} = \frac{1}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}$$

To find the appropriate values for A, B, we combine the two terms in a single fraction with denominator $x^2 - 4$ and demand that the equality is satisfied for any x. Explicitly,

$$\frac{1}{x^2-4} = \frac{A}{x-2} + \frac{B}{x+2} = \frac{Ax + 2A + Bx - 2B}{x^2-4} = \frac{x(A+B) + 2(A-B)}{x^2-4}$$

Demanding that the equality is true for any value of x we obtain

$$A + B = 0$$
$$2(A - B) = 1$$

Solving the system of equations for A, B we find that $A = \frac{1}{4}$, $B = -\frac{1}{4}$, therefore

$$\frac{1}{x^2-4} = \frac{1}{4}\frac{1}{x-2} - \frac{1}{4}\frac{1}{x+2}$$

We can now use this equality to solve the integral, i.e.,

$$\int \frac{dx}{x^2-4} = \int \frac{1}{4}\ \frac{1}{x-2}dx - \int \frac{1}{4}\ \frac{1}{x+2}dx = \frac{1}{4}\ln|x-2| - \frac{1}{4}\ln|x+2| + C \Rightarrow$$
$$\int \frac{dx}{x^2-4} = \frac{1}{4}\ln|\frac{x-2}{x+2}| + C$$

Question 1.3

$$\int \sinh^8 x \cosh x dx$$

Solution:
This integral is simply solved with a change of variable. Let us set:

$$u = \sinh x \Rightarrow du = \cosh x dx$$

to express the original integral as follows:

$$\int \sinh^8 x \cosh x dx = \int u^8 du = \frac{u^9}{9} + C = \frac{\sinh^9 x}{9} + C$$

Question 1.4

$$\int \sin x \cos^7 x dx$$

Solution:
This integral is similar to the previous one, the only difference being that the hyperbolic sin, cos functions have been replaced by the trigonometric counterparts. Notice also that now it is the trigonometric cos function that is raised to the power 7. This fact guides us to the following substitution:

$$u = \cos x \Rightarrow du = -\sin x dx$$

which enables us to write the original integral as follows:

$$\int \sin x \cos^7 x dx = -\int u^7 du = -\frac{u^8}{8} + C = -\frac{\cos^8 x}{8} + C$$

Question 1.5

$$\int_1^2 dx x^5 \ln x$$

Solution:
To solve this integral we simply need to integrate by parts:

$$\int_1^2 dx x^5 \ln x = \int_1^2 dx (\frac{x^6}{6})' \ln x = \frac{x^6}{6} \ln x \Big|_1^2 - \int_1^2 \frac{x^6}{6} (\ln x)' \, dx = \frac{2^6 \ln 2}{6} - \int_1^2 \frac{x^6}{6} \frac{1}{x} =$$
$$= \frac{32 \ln 2}{3} - \int_1^2 \frac{x^5}{6} = \frac{32 \ln 2}{3} - \frac{1}{6} \frac{x^6}{6} \Big|_1^2 = \frac{32 \ln 2}{3} - \frac{1}{36} \left(2^6 - 1\right) = \frac{32 \ln 2}{3} - \frac{7}{4}$$

Question 1.6

$$\int_0^1 \frac{8x+6}{\sqrt{3x+2x^2}} dx$$

Solution:
Notice some similarity between the numerator and the denominator; the numerator is actually a multiple of the derivative of the expression under the square root in the denominator. This implies that we can write the integrand as a total derivative. In fact,

$$(\sqrt{2x^2+3x})' = \frac{4x+3}{2\sqrt{2x^2+3x}}$$

so the integrand is equal to four times the above expression, i.e.,

$$\int_0^1 \frac{8x+6}{\sqrt{3x+2x^2}} dx = \int_0^1 \left(4\sqrt{2x^2+3x}\right)' dx = 4\sqrt{2x^2+3x}\Big|_0^1 = 4\sqrt{5}$$

Question 1.7

$$\int xe^{-x}dx$$

Solution:
This is one of the simplest integrals that can be solved with integration by parts. We only need to be careful with the signs.

$$\int xe^{-x}dx = \int x(-e^{-x})'\,dx = -xe^{-x} - \int (-e^{-x})(x)'\,dx = -xe^{-x} - \int (-e^{-x})\,dx =$$
$$= -xe^{-x} + \int e^{-x} = -xe^{-x} - e^{-x} + C$$

Question 1.8

$$\int \frac{9ydy}{2y^2+3}$$

Solution:
To compute this integral we change variable by setting

$$u = y^2 \Rightarrow du = 2ydy \Leftrightarrow du\frac{1}{2} = ydy$$

and rewritting the original integral as follows:

$$\int du\frac{1}{2}\frac{9}{2u+3} = \int du\frac{1}{2}\,\frac{9}{2(u+\frac{3}{2})} = \int du\frac{9}{4}\,\frac{1}{u+\frac{3}{2}} = \frac{9}{4}\ln\left|u+\frac{3}{2}\right| + C = \frac{9}{4}\ln\left|y^2+\frac{3}{2}\right| + C$$

Question 1.9

$$\int \frac{dx}{x^4-1}$$

Solution:
This is another integral which can be solved by expressing the integrand in terms of partial fractions. Specifically, we assume the following equality

$$\frac{1}{x^4-1} = \frac{1}{(x-1)(x+1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}\,,$$

and solve for the constants A, B, C, D. Concretely,

$$\frac{1}{x^4-1} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1} = \frac{A(x+1)(x^2+1) + B(x-1)(x^2+1) + (Cx+D)(x^2-1)}{x^4-1} =$$
$$= \frac{x^3(A+B+C) + x^2(A-B+D) + x(A+B-C) + (A-B-D)}{x^4-1}\,.$$

For the equality above to be satisfied for any x it must be that:

$$\begin{aligned} A+B+C&=0\\ A-B+D&=0\\ A+B-C&=0\\ A-B-D&=1 \end{aligned}$$

This system of four equations with four unkowns is easily solvable; the solution is

$$A=\frac{1}{4}, B=-\frac{1}{4}, C=0, D=-\frac{1}{2},$$

leading to

$$\frac{1}{x^4-1}=\frac{1}{4}\frac{1}{x-1}-\frac{1}{4}\frac{1}{x+1}-\frac{1}{2}\frac{1}{x^2+1}.$$

Evaluating the integral is now trivial;

$$\int dx\frac{1}{x^4-1}=\frac{1}{4}\int dx\frac{1}{x-1}-\frac{1}{4}\int dx\frac{1}{x+1}-\frac{1}{2}\int dx\frac{1}{x^2+1}=\frac{1}{4}\ln|x-1|-\frac{1}{4}\ln|x+1|-\frac{1}{2}\arctan x+C\Rightarrow$$

$$\Rightarrow\int dx\frac{1}{x^4-1}=\frac{1}{4}\ln\left|\frac{x-1}{x+1}\right|-\frac{1}{2}\arctan x+C$$

Notice that we used the following identity:

$$\int dx\frac{1}{x^2+1}=\arctan x+C$$

If you do not remember this identity, you can prove it with the method of trigonometric substitution. Namely, by defining $x=\tan u\Rightarrow dx=\frac{du}{\cos^2 u}$ and using the trigonometric identity

$$\frac{1}{\cos^2 u}=1+\tan^2 u.$$

Question 1.10

$$\int\frac{\ln^3 x}{x}dx$$

Solution:
To solve this integral we define a new variable u according to:

$$u=\ln x\Rightarrow du=\frac{dx}{x}$$

and rewrite the integral as follows:

$$\int\frac{\ln^3 x}{x}dx=\int u^3du=\frac{u^4}{4}+C=\frac{\ln^4 x}{4}+C$$

Question 1.11

$$\int \frac{x^2}{(x-1)(x-2)}dx$$

Solution:
Let us first rewrite the integrand as follows:

$$\begin{aligned}\frac{x^2}{(x-1)(x-2)} &= \frac{x^2-1+1}{(x-1)(x-2)} = \frac{x^2-1}{(x-1)(x-2)} + \frac{1}{(x-1)(x-2)} = \\ &= \frac{(x-1)(x+1)}{(x-1)(x-2)} + \frac{1}{(x-1)(x-2)} = \frac{x+1}{(x-2)} + \frac{1}{(x-1)(x-2)} = \\ &= \frac{x-2+3}{(x-2)} + \frac{1}{(x-1)(x-2)} = 1 + 3\frac{1}{(x-2)} + \frac{1}{(x-1)(x-2)}\,.\end{aligned}$$

Using this identity, we can break up the original integral into three separate integrals which can be easily solved:

$$\int \frac{x^2}{(x-1)(x-2)}dx = \int dx + \int 3\,\frac{1}{(x-2)}dx + \int \frac{1}{(x-1)(x-2)}dx$$

The first two integrals are trivial, while the last one can be evaluated by decomposing the integrand into partial fractions. Specifically,

a.

$$\int dx = x + C_1$$

b.

$$\int 3\,\frac{1}{(x-2)}dx = 3\ln|x-2| + C_2$$

c.

$$\int \frac{1}{(x-1)(x-2)} = -\int \frac{1}{x-1}dx + \int \frac{1}{x-2} = -\ln|x-1| + \ln|x-2| + C_3\,.$$

Since we have explicitly shown how to decompose similar expressions into partial fractions in other examples, we did not go through the details here but rather directly wrote the appropriate decomposition. Notice also, that each constant is in principle different from the other one - this is why we used a subindex to distinguish between them.

Substituting the results obtained by performing the integrals (a,b,c) we arrive at:

$$\int \frac{x^2}{(x-1)(x-2)}dx = x + 4\ln|x-2| - \ln|x-1| + C\,,$$

where we set $C = C_1 + C_2 + C_3$ for simplicity.

Question 1.12

$$\int \frac{10dx}{x^2+9}$$

Solution:
Here it is useful to recall the identity used in **Question 1.9**, i.e.,

$$\int dx \frac{1}{x^2+1} = \arctan x + C\,.$$

With this in mind, we set $u = \frac{x}{3} \Rightarrow du = \frac{1}{3}dx \Rightarrow 3du = dx$ to obtain

$$\int dx \frac{10}{x^2+9} = \int dx\, \frac{10}{9(\frac{x^2}{9}+1)} = \int du\, 3\, \frac{10}{9}\, \frac{1}{u^2+1} = \frac{10}{3} \arctan u + C = \frac{10}{3} \arctan \frac{x}{3} + C$$