# Calculus for Engineers II - Sample Problems on mathematical modeling with differential equations.

**Exercise 1:** Assume that some population of bacteria grows with time according to the logistic growth model, i.e.,

$$y = \frac{L}{1 + Ae^{-kt}}$$

where $y(t)$ is the population at time $t \geq 0$ and $A, k, L$ are some positive constants.

1. What is the rate of growth of the population at time $t = t_0 > 0$?
2. What is the rate of growth of the population at time $t = 0$ ?
3. What is the initial, at time $t = 0$, value of the population?
4. What would be the value of the population after an infinite amount of time has passed?

**Solution:** To find the rate of growth of the population at some arbitrary time $t$, we need to compute the derivative $y'(t)$.

$$\begin{aligned} y'(t) &= L((1 + Ae^{-kt})^{-1})' = L(-1)(1 + Ae^{-kt})^{-2}(1 + Ae^{-kt})' = \\ &= L(-1)(1 + Ae^{-kt})^{-2}Ae^{-kt}(-kt)' = L(-1)(1 + Ae^{-kt})^{-2}Ae^{-kt}(-k) = \\ &= \frac{LAke^{-kt}}{(1 + Ae^{-kt})^2} \end{aligned}$$

Once we have the rate of growth at arbitrary times $t$ we can evaluate it at time $t = t_0$ to find the rate of growth at $t = t_0$:

$$y'(t = t_0) = \frac{LAke^{-kt_0}}{(1 + Ae^{-kt_0})^2}$$

In a similar manner, we obtain the rate of growth at time $t = 0$,

$$y'(t = 0) = \frac{LAk}{(1 + A)^2}$$

The initial value of the population is

$$y(t = 0) = \frac{L}{1 + Ae^{-k0}} = \frac{L}{1 + A}$$

whereas the value of the population after an infinite amount of time has passed is equal to

$$\lim_{t \to +\infty} y(t) = \lim_{y \to \infty} \frac{L}{1 + Ae^{-kt}} = L$$

where we used the fact that $\lim_{x\to=\infty} e^{-x} = 0$.

**Exercise 2**: Consider a model describing the decay of some population $y$ according to the differential equation

$$\frac{dy}{dt} = -ky$$

where $k$ is some positive constant. Assume that the population is equal to $y = a$ at time $t = t_0$ and

1. Find a formula for the growth of the population $y(t)$ for time $t > t_0$ by solving the above differential equation .

2. Find the time for which the population is equal to half its value at time $t = t_0$.

**Solution:** The differential equation for the decay of the population can be solved by separating variables:

$$\frac{dy}{dt} = -ky \Leftrightarrow \frac{dy}{y} = -kdt \Rightarrow \int \frac{dy}{y} = -k \int dt \Leftrightarrow$$
$$\Leftrightarrow \ln y = -kt + C \Leftrightarrow y(t) = e^{-kt}e^{C} \Rightarrow y(t) = ce^{-kt}$$

where in the last line we introduced a new constant $c$ by defininig $c = e^C$ for convenience.

To specify the value of $c$ we must use the initial data given in the problem, i.e., that the population at time $t = t_0$ is equal to $y(t = t_0) = a$.

$$y(t_0) = a = ce^{-kt_0} \Rightarrow c = ae^{kt_0}$$

Substituting the above value of the constant $c$ into the solution for $y(t)$ we obtain

$$y(t) = ae^{-k(t-t_0)}$$

We are now ready to move to the next part of this problem. To find the time $T$ for which the population is equal to half its value at time $t = t_0$, i.e., $y(t = T) = \frac{a}{2}$ we simply use the solution for $y(t)$ obtained above.

$$y(t = T) = \frac{a}{2} = ae^{-k(T-t_0)} \Rightarrow \frac{1}{2} = e^{-k(T-t_0)} \Leftrightarrow \ln\frac{1}{2} = -k(T - t_0) \Leftrightarrow$$
$$\Leftrightarrow -\ln 2 = -kT + kt_0 \Leftrightarrow T = t_0 + \frac{1}{k}\ln 2$$

**Exercise 3** A bullet of mass $m$, fired straight up with an initial velocity $v_0$, is slowed down by the force of gravity $mg$ and a drag force of air resistance $kv^2$, where $g$ is the gravitational constant of acceleration and $k$ a positive constant. As the bullet moves upward, its velocity $v$ satisfies the equation

$$m\frac{dv}{dt} = -(kv^2 + mg)$$

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1. Show that if $x = x(t)$ is the height of the bullet above the barrel opening at time t, and $v = \frac{dx}{dt}$, then
$$mv\frac{dv}{dx} = -(kv^2 + mg)$$
2. Solve the above differential equation to find $x$ as a function of $v$, given that the initial velocity of the bullet, for $x = 0$, is $v = v_0$.
3. Using your result in part (2) find the maximum height the bullet reaches. Use the fact that the maximum height is attained when the velocity $v$ vanishes.

**Solution:** Using the chain rule, we can rewrite

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

Substituting to the given differential equation we find that

$$m\frac{dv}{dt} = -(kv^2 + mg) = mv\frac{dv}{dx} = -(kv^2 + mg)$$

Having answered **E3.1** we proceed to solve the differential equation for $x(v)$.

$$mv\frac{dv}{dx} = -(kv^2 + mg) \Leftrightarrow \frac{mvdv}{(kv^2 + mg)} = -dx \Rightarrow m\int \frac{vdv}{(kv^2 + mg)} = -\int dx$$

The integration can be easily performed by substituting $u = v^2 \Rightarrow du = 2vdv$

$$\begin{aligned} \frac{m}{2}\int \frac{du}{ku + mg} = -\int dx \Leftrightarrow \frac{m}{2k}\int \frac{du}{u + \frac{mg}{k}} = -\int dx \Leftrightarrow \\ \frac{m}{2k}\ln\left(u + \frac{mg}{k}\right) = -x + C \Leftrightarrow \frac{m}{2k}\ln\left(v^2 + \frac{mg}{k}\right) = -x + C \Leftrightarrow \\ x = C - \frac{m}{2k}\ln\left(v^2 + \frac{mg}{k}\right) \end{aligned}$$

The final step is to fix the constant $C$ by using the fact that when $x = 0$, $v = v_0$. Substituting into the last equality we find that

$$0 = C - \frac{m}{2k}\ln\left(v_0^2 + \frac{mg}{k}\right) \Rightarrow C = \frac{m}{2k}\ln\left(v_0^2 + \frac{mg}{k}\right)$$

so that

$$x = \frac{m}{2k} \ln \left( \frac{v_0^2 + \frac{mg}{k}}{v^2 + \frac{mg}{k}} \right)$$

We now move on to answer **E3.3**. The statement of the problem informs us that the bullet reaches its maximum height when its velocity becomes zero, i.e. $h_{max} = h(v = 0)$. We only then need to substitute $v = 0$ in the expression for $x(v)$:

$$x_{max} = \frac{m}{2k} \ln \left( \frac{v_0^2 + \frac{mg}{k}}{0 + \frac{mg}{k}} \right) = \frac{m}{2k} \ln \left( \frac{kv_0^2 + mg}{mg} \right)$$

**Exercise 4:** Suppose that a tank containing a liquid is vented to the air at the top and has an outlet at the bottom through which the liquid can drain. According to Torricelli's law, if the outlet is opened at time $t = 0$, then at each instant the depth of the liquid $f(t)$ inside the tank and the area $A(f)$ of the "free surface" of the liquid, are related by

$$A(f)\frac{df}{dt} = -k\sqrt{f}$$

where $k$ is a positive constant which depends on such factors as the viscosity of the liquid, or the cross-sectional area of the outlet.

**Exercise 4a:** Suppose that the cylindrical tank in figure 1. is full with liquid at time $t = 0$, and the total height $h$ of the tank, the radius $r$ of the circular top of the cylinder, as well as the constant $k$ in Torricelli's law are given. Then:

1. Find f(t).
2. Find the time that is needed for the liquid to drain completely.

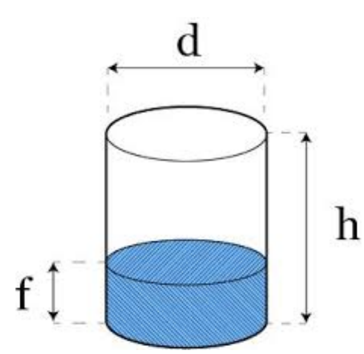


Figure 1: The cylindrical tank of E4a.

**Solution:** According to figure 1. the free surface of the liquid does not change as the liquid drains. The area of that surface is equal to $A(f) = \pi r^2$, where $r$ is the radius of the top of the cylinder. Substituting the expression for the area in Torricelli's law we proceed to slve the diffrential equation for $f(t)$ by separating variables:

$$\pi r^2 \frac{df}{dt} = -k\sqrt{f} \Leftrightarrow \pi r^2 \frac{df}{\sqrt{f}} = -kdt \Rightarrow \int \frac{df}{\sqrt{f}} = -\frac{k}{\pi r^2}\int dt \Leftrightarrow 2\sqrt{f} = -\frac{k}{\pi r^2}t + C \Rightarrow$$
$$\Rightarrow f(t) = \left(-\frac{k}{2\pi r^2}t + \frac{C}{2}\right)^2$$

To determine the value of the constant $C$, recall that at time $t = 0$ the tank was full of liquid. Therefore, the height of the liquid $f(t = 0) = h$. Substituting into the resulting expression for $f(t)$ we obtain:

$$h = \left(\frac{C}{2}\right)^2 \Leftrightarrow \frac{C}{2} = \pm\sqrt{h}$$

We choose the positive sign solution for $C$ because we know that the height of the liquid $f$ gets reduced with time. The answer to question **E4a.1** is thus:

$$f(t) = \left(\sqrt{h} - \frac{k}{2\pi r^2}t\right)^2$$

Once we have an expression for the height of liquid as a function of time we can proceed to answer **E4a.2**. The time $t = T$ for which the tank will be empty from the liquid, is the time for which $f(t = T) = 0$, i.e.,

$$0 = \left(\sqrt{h} - \frac{k}{2\pi r^2}T\right)^2 \Rightarrow T = \frac{\sqrt{h}2\pi r^2}{k}$$

**Exercise 4b:** Answer the questions of **E4a1.** if the cylindrical tank is positioned as in figure 2, and at time $t = 0$ is completely full of liquid.

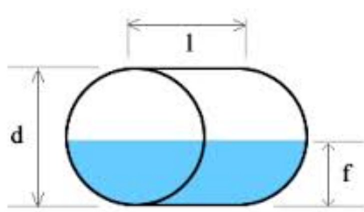


Figure 2: The cylindrical tank associated to E4b.

**Solution:** The difference from **E4a** lies in the geometry indicated in the picture. In this case, the area of the free surface of the liquid will change with time as the liquid gets drained from the tank. We need to understand how the area will change and express its change as a function of the total height/depth of the liquid $f$.

First, we observe that the area of the free surface of the liquid is the area of a rectangle. The area of a rectangle is equal to the product of its two different sizes. One size of the rectangle is equal to the total length $\ell$ of the cylindrical tank and will not change with time. The other size, will change as the liquid flows out of the tank. Let us denote this size with $b$ and assume that at time $t$ it is as in figure 3. Observe that we can relate $b$ by the Pythagorean theorem to the radius of the top of the cylinder as follows:

$$\left(\frac{b}{2}\right)^2 + (f-r)^2 = r^2 \Leftrightarrow b = 2\sqrt{r^2-(f-r)^2} = 2\sqrt{2fr-f^2}$$

where $f$ is the height of the liquid. The area of the free surface is then $A(f) = 2\ell\sqrt{f(2r-f)}$.

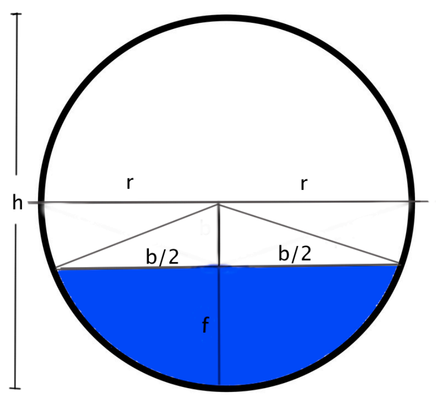


Figure 3: How is $b$ related to $f(t)$?

We can now substitute $A(f)$ into Torricelli's law and solve for $f(t)$.

$$2\ell\sqrt{f(2r-f)}\frac{df}{dt} = -k\sqrt{f} \Leftrightarrow \sqrt{f}\sqrt{2r-f}\frac{df}{\sqrt{f}} = -\frac{k}{2\ell}dt \Leftrightarrow \int\sqrt{2r-f}df = -\frac{k}{2\ell}\int dt \Leftrightarrow$$
$$-\int\sqrt{2r-f}d(2r-f) = -\frac{k}{2\ell}\int dt \Leftrightarrow -\frac{2}{3}(2r-f)^{\frac{3}{2}} = -\frac{k}{2\ell}t + C \Rightarrow f(t) = 2r - \left(\frac{3k}{4\ell}t - \frac{3C}{2}\right)^{\frac{2}{3}}$$

The constant $C$ can be specified from the fact that the tank is full of liquid at time $t=0$, thus $f(t=0)=2r$.

$$2r = 2r - \left(-\frac{3C}{2}\right)^{\frac{2}{3}} \Rightarrow C = 0$$

Substituting the value $C = 0$ into the solution of the differential equation we obtain an expression for the height of the liquid as a function of time

$$f(t) = 2r - \left(\frac{3k}{4\ell}\right)^{\frac{2}{3}} t^{\frac{2}{3}}$$

We proceed to find the time $t = T$ for which the tank will contain no liquid , i.e., $f(T) = 0$.

$$0 = 2r - \left(\frac{3k}{4\ell}\right)^{\frac{2}{3}} T^{\frac{2}{3}} \Leftrightarrow T = \frac{4\ell}{3k}(2r)^{\frac{3}{2}}$$

**Note:** Some of the exercises here are selected from the WileyPlus: "Additional Material: Mathematical Modeling with Differential Equations".