Topological Gauge Theory, Quivers and Flow Trees

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Anna, Tom, Andrea and Kento:

On behalf of all participants, thanks very much for organizing:

BPS states, mirror symmetry, and exact WKB

28 June--2 July 2021

This talk is based on arXiv:2004.14466, joint work with G. Beaujard and B. Pioline

Topological Gauge Theory

Topological gauge theory is of interest physically as a setting in which many aspects of quantum field theory can be analyzed exactly.
At the same time, topological gauge theory connects to various subjects in mathematics such as (algebraic) geometry, modular forms etc.
Today, I will consider the topological twist of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory known as Vafa-Witten theory on Fano four-manifolds. I will explain how low order terms of the partition functions can be determined using quivers.

Field content of VW theory

Let X be a smooth, compact four-manifold. Let P be a principle G-bundle.

The bosonic fields in the twisted theory are:

- Field strength $F \in \Omega^2(X, \mathrm{ad} P)$
- Complex scalar field $\phi \in \Omega^0(X, \mathrm{ad} P) \otimes \mathbb{C}$
- Real scalar field $C \in \Omega^0(X, \mathrm{ad} P)$
- Self-dual 2-form $B^+ \in \Omega^{2+}(X, \mathrm{ad} P)$

Q-fixed equations

The Q-fixed equations are

$$F^+ + [C, B^+] + [B^+ . B^+] = 0$$

$$dC + d * B^+ = 0$$

In general, these equations have multiple components of solutions. However, for constant scalar curvature $R \geq 0$, one has

$$C = B^+ = 0 \Rightarrow F^+ = 0$$

I.e., only the instanton component is non-empty. These solutions saturate the BPS inequality

$$\int_X \mathrm{Tr}[F \wedge *F] \geq \left| \int_X \mathrm{Tr}[F \wedge F] \right|$$

$U(N)$ Instantons are famously in 1-to-1 correspondence with semi-stable vector bundles. Let $\gamma \in (N, c_1, \mathrm{ch}_2) = (N, c_1, \frac{1}{2}c_1^2 - c_2)$ be the Chern character of the gauge bundle, and $\mathcal{M}(\gamma; J)$ the moduli space of Gieseker semi-stable vector bundles wrt to the polarization J.

Partition function

$\mathcal{N} = 4$ YM is a superconformal theory with UV coupling constant $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$ Let

$$q = e^{2\pi i \tau}$$

The partition function of the theory is a generating function of Euler numbers. Let us fix $N = 2$, $\mu = c_1$, $n = \mathrm{ch}_2$, and

$$c_\mu(n) = \chi(\mathcal{M}(\gamma; J))$$

Then,

$$h_\mu(\tau) = \sum_n c_\mu(n)\, q^n$$

For the purpose of this talk, I restrict to $X = \mathbb{P}^2$. All discussion continues with minor modifications to other toric four-manifolds, for example $\mathbb{F}_0$.
The mirror of the non-compact CY $K_{\mathbb{F}_0}$ is a local curve. See the talk by Pietro Longhi in this workshop for the relation to spectral and exponential networks.

Partition functions for $N = 2$

A combination of results by Klyachko, Vafa-Witten, Yoshioka gives for $X = \mathbb{P}^2$

$$h_\mu(\tau) = \frac{f_\mu(\tau)}{q^{1/4}\prod_{n=1}^\infty (1-q^n)^6}$$

with f_μ generating functions of Hurwitz class numbers $H(n)$

$$f_0(\tau) = \sum_{n=0}^{\infty} 3H(4n)\, q^n = -\frac{1}{4} + \frac{3}{2}q + 3q^2 + \dots,$$

and

$$f_1(\tau) = \sum_{n=1}^{\infty} 3H(4n-1)\, q^{n-1/4} = q^{3/4}(1 + 3q + 3q^2 + \dots),$$

Partition functions for $N = 2$

- Yoshioka's approach was originally based on working over finite fields $\mathbb{F}_q$ and the Weil conjectures.
- f_μ are examples of mock modular forms studied by Ramanujan, Zagier, Zwegers...

Partition functions for $N > 2$

Generalizing the approach of Yoshioka combined with wall-crossing formula's, generating functions of Poincaré polynomial,

$$p(y;\mathcal{M}) = \sum_{j=0}^{\dim_{\mathbb{C}} X} b_j(\mathcal{M})\, y^j$$

of $\mathcal{M}(\gamma; J)$ can be determined for arbitrary γ and J. JM 2011-18

These are generalizations of mock modular forms, ie mock modular forms of depth $N-1$. The Darboux coordinates $\mathcal{X}_\gamma$ discussed in Neitzke's talk were instrumental for derivations of the non-holomorphic part

Alexandrov, Banerjee, JM, Pioline '16-'17, Alexandrov, JM Pioline '19

From arXiv:2004.14466:

$$\begin{aligned}
h^{\mathbb{P}^2}_{5,0} &= \frac{1}{B_{5,0}}\left[H_5 + 2\Psi^{0,0}_{1,4}\,H_1\,H_4 + 2\Psi^{0,0}_{3,2}\,H_2\,H_3 + \left(2\Psi^{0,0}_{2,2,1} + \Psi^{0,0}_{2,1,2}\right)\,H_2^2 H_1\right.\\
&\quad \left. + \left(2\Psi^{0,0}_{2,1,1,1} + 2\Psi^{0,0}_{1,2,1,1}\right) H_1^3\,H_2 + \left(2\Psi^{0,0}_{3,1,1} + \Psi^{0,0}_{1,3,1}\right) H_1^2\,H_3 + \Psi^{0,0}_{1,1,1,1,1} H_1^5\right]\\
&\quad -\frac{1}{120}(h^{\mathbb{P}^2}_1)^5 - \frac{1}{6}h^{\mathbb{P}^2}_{2,0}(h^{\mathbb{P}^2}_1)^3 - \frac{1}{2}h^{\mathbb{P}^2}_{3,0}(h^{\mathbb{P}^2}_1)^2 - \frac{1}{2}(h^{\mathbb{P}^2}_{2,0})^2 h^{\mathbb{P}^2}_1 - h^{\mathbb{P}^2}_{4,0}h^{\mathbb{P}^2}_1 - h^{\mathbb{P}^2}_{2,0}h^{\mathbb{P}^2}_{3,0},\\
h^{\mathbb{P}^2}_{5,1} &= \frac{1}{B_{5,0}}\left[H_5 + 2\Psi^{-1,1}_{1,4}\,H_1\,H_4 + 2\Psi^{-1,1}_{3,2}\,H_2\,H_3 + \left(2\Psi^{-1,1}_{2,2,1} + \Psi^{-1,1}_{2,1,2}\right)\,H_2^2 H_1\right.\\
&\quad \left. + \left(2\Psi^{-1,1}_{2,1,1,1} + 2\Psi^{-1,1}_{1,2,1,1}\right) H_1^3\,H_2 + \left(2\Psi^{-1,1}_{3,1,1} + \Psi^{-1,1}_{1,3,1}\right) H_1^2\,H_3 + \Psi^{-1,1}_{1,1,1,1,1} H_1^5\right],\\
h^{\mathbb{P}^2}_{5,2} &= \frac{1}{B_{5,0}}\left[H_5 + 2\Psi^{-2,2}_{1,4}\,H_1\,H_4 + 2\Psi^{-2,2}_{3,2}\,H_2\,H_3 + \left(2\Psi^{-2,2}_{2,2,1} + \Psi^{-2,2}_{2,1,2}\right)\,H_2^2 H_1\right.\\
&\quad \left. + \left(2\Psi^{-2,2}_{2,1,1,1} + 2\Psi^{-2,2}_{1,2,1,1}\right) H_1^3\,H_2 + \left(2\Psi^{-2,2}_{3,1,1} + \Psi^{-2,2}_{1,3,1}\right) H_1^2\,H_3 + \Psi^{-2,2}_{1,1,1,1,1} H_1^5\right],
\end{aligned} \tag{A.43}$$

From arXiv:2004.14466:

$$
\begin{aligned}
\mathfrak{h}_{5,0}^{\mathbb{P}^2} &= \mathrm{q}^{5/8}(y-1/y)\left[h_{5,0}^{\mathbb{P}^2} - \frac{1}{5}h_1(5\tau, 5w)\right] \\
&= \mathrm{q}^5\left(y^{26} + y^{24} + 3y^{22} + 5y^{20} + 9y^{18} + 13y^{16} + 18y^{14} + 22y^{12} + 26y^{10}\right. \\
&\quad \left. + 28y^8 + 30y^6 + 30y^4 + 31y^2 + 31 + \dots\right) \\
&\quad + \mathrm{q}^6\left(y^{36} + 2y^{34} + 5y^{32} + 10y^{30} + 20y^{28} + 35y^{26} + 61y^{24} + 96y^{22} + 148y^{20} + 212y^{18}\right. \\
&\quad \left. + 289y^{16} + 368y^{14} + 446y^{12} + 509y^{10} + 561y^8 + 596y^6 + 620y^4 + 632y^2 + 638 + \dots\right) \\
&\quad + \dots \\
\mathfrak{h}_{5,1}^{\mathbb{P}^2} &= \mathrm{q}^4\left(y^{12} + y^{10} + 3y^8 + 5y^6 + 8y^4 + 10y^2 + 12 + \dots\right) + \mathrm{q}^5\left(y^{22} + 2y^{20} + 5y^{18}\right. \\
&\quad \left. + 10y^{16} + 20y^{14} + 34y^{12} + 57y^{10} + 87y^8 + 126y^6 + 165y^4 + 198y^2 + 210 + \dots\right) \\
&\quad + \mathrm{q}^6\left(y^{32} + 2y^{30} + 6y^{28} + 12y^{26} + 26y^{24} + 48y^{22} + 89y^{20} + 150y^{18} + 251y^{16} + 393y^{14}\right. \\
&\quad \left. + 600y^{12} + 865y^{10} + 1201y^8 + 1564y^6 + 1921y^4 + 2177y^2 + 2280 + \dots\right) + \dots \\
\mathfrak{h}_{5,2}^{\mathbb{P}^2} &= \mathrm{q}^4 + \mathrm{q}^5\left(y^{10} + 2y^8 + 5y^6 + 8y^4 + 13y^2 + 14 + \dots\right) + \mathrm{q}^6\left(y^{20} + 2y^{18} + 6y^{16}\right. \\
&\quad \left. + 12y^{14} + 25y^{12} + 44y^{10} + 76y^8 + 114y^6 + 161y^4 + 196y^2 + 214 + \dots\right) + \dots . \qquad \text{(A.44)}
\end{aligned}
$$

Besides Yoshioka's approach, there is another approach to bundles on $\mathbb{P}^2$, namely using quivers or monads. This goes back to work by Beilinson, Drézet, le Potier,…from 1970's, 80's, and work on D-branes by Douglas, Fiol, Rommelsberger for $\mathbb{C}^3/\mathbb{Z}_3$,…. The resolution of $\mathbb{C}^3/\mathbb{Z}_3$ is the total space of the canonical bundle $K_{\mathbb{P}^2}$. This geometry appears in the geometric engineering of the rank 1 E_0 5d SCFT. See for example Morrison, Seiberg (1996),…, Closset, Del Zotto (2019)

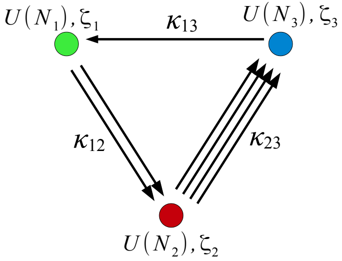
Yet other approaches are:

- toric localization Klyachko (1991), Kool (2009), Bonelli *et al* (2021), and others,
- the u-plane JM, Moore (2021).
- …

Quiver

Quiver:

1. Vertices $i \in V$
2. Gauge groups $U(N_i)$
3. FI parameters ζ_i, $\sum_i \zeta_i = 0$
4. $|\kappa_{ij}|$ bifundamental chiral multiplets ϕ^a_{ij}, $a = 1, \ldots, |\kappa_{ij}|$
5. Superpotential $W(\phi^a_{ij})$



Quiver moduli space

D-term equations:

$$\zeta_i = \sum_{a,j,\kappa_{ij}>0} (\phi^a_{ij})^\dagger \phi^a_{ij} - \sum_{a,j,\kappa_{ij}<0} (\phi^a_{ij})^\dagger \phi^a_{ij}$$

F-term equations:

$$\frac{\partial W(\phi^a_{ij})}{\partial \phi^b_{k\ell}} = 0, \qquad \forall b, k, \ell$$

Quiver moduli space $\mathcal{M}^Q(\vec{N}; \vec{\zeta})$:
Space of ϕ^a_{ij} satisfying D- and F-term equations

BPS index:

$$\Omega(\vec{N}, \vec{\zeta}) = \chi(\mathcal{M}^Q(\vec{N}; \vec{\zeta}))$$

In favourable circumstances, D- and F-term conditions are solved by a "cut",

$$\phi^b_{i'j'} = 0$$

for specific pair(s) $i'j'$, and then imposing the equations

$$\frac{\partial W(\phi^a_{ij})}{\partial \phi^b_{i'j'}} = 0, \qquad b = 1, \ldots, |\kappa_{i'j'}|$$

Then the virtual dimension can be determined as

$$\dim(\mathcal{M}^Q(\vec{N}, \vec{\zeta})) = \sum_{ij \neq i'j'} |\kappa_{ij}| N_i N_j - \sum_{i'j'} |\kappa_{i'j'}| N_{i'} N_{j'} - \sum_i N_i^2 + 1$$

This depends on the superpotential, and is related to perfect matchings of the associated brane tiling.

Attractor stability

The "attractor stability" for a dimension vector $\vec{N} = (N_1, N_2, \dots)$ is

$$\zeta_i^a = -\kappa_{ij} \, N^j$$

This follows from the supergravity approach to BPS quivers.

Mathematically, this stability conditions is known as "self-stability", Bridgeland (2016)

Exceptional collections provide a way to associate a BPS quiver to an algebraic surface.

Exceptional collections

A sheaf E on X is *exceptional* if

$$\mathrm{Ext}^0(E,E) \cong \mathbb{C}, \qquad \mathrm{Ext}^k(E,E) = 0, \quad k > 0$$

An *exceptional collection* on X is an ordered set of exceptional objects $\mathcal{C} = (E^1, \dots, E^r)$, such that

$$\mathrm{Ext}^k(E^i, E^j) = 0, \quad \forall k \geq 0, 1 \leq j < i \leq r$$

An exceptional collection is *full*, if $\mathrm{ch}(E^i)$ generate $K(X)$
An exceptional collection is *strong*, if $\mathrm{Ext}^k(E^i, E^j) = 0$ for all i, j and $k > 0$

Exceptional collections

Not all surfaces gives rise to exceptional collections. Strong, full exceptional collections are known for

- toric surfaces, including $\mathbb{P}^2$
- del Pezzo surfaces dP_k, $1 \leq k \leq 8$
- pseudo del Pezzo surfaces PdP_k, $1 \leq k \leq 6$

Exceptional collection and quivers

To determine the quiver, one determines a dual collection $\mathcal{C}^\vee = (E_1^\vee, \ldots, E_r^\vee)$ with

$$\chi(E^i, E_j^\vee) = \delta^i_j$$

The dual collection is exceptional and full, but not strong.

Quiver:

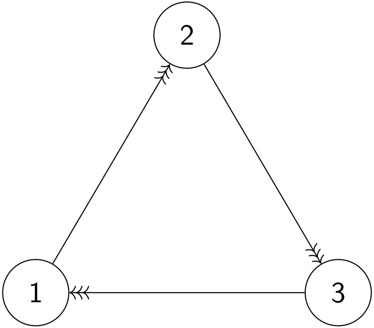
- The nodes correspond to the elements of $\mathcal{C}^\vee$
- (Net) number of arrows:

$$\kappa_{ij} = \langle \gamma_i, \gamma_j \rangle = \int_X (\mathrm{ch}((E_j^\vee)^*)\mathrm{ch}(E_i^\vee) - \mathrm{ch}((E_i^\vee)^*)\mathrm{ch}((E_j^\vee)))\,\mathrm{Td}(X)$$

- Superpotential follows from algebraic-geometric or string-theoretic techniques

Aspinwall, Katz (2004), Hanany, Seong (2012), ...

Quiver for $\mathbb{P}^2$



with potential

$$W = \sum_{(\alpha\beta\gamma)\in S_3} \operatorname{sgn}(\alpha\beta\gamma)\, \phi_{12}^{\alpha}\phi_{23}^{\beta}\phi_{31}^{\gamma}$$

Quiver for $\mathbb{P}^2$

The standard strong collection is $\mathcal{C} = (\mathcal{O}, \mathcal{O}(H), \mathcal{O}(2H))$, which leads to the following Chern characters for the dual collection $E_i^\vee$:

$$\begin{aligned}
\gamma_1 &= (1,0,0) \in H^0(\mathbb{P}^2) \oplus H^2(\mathbb{P}^2) \oplus H^4(\mathbb{P}^2) \\
\gamma_2 &= (-2,1,1/2) \\
\gamma_3 &= (1,-1,1/2)
\end{aligned}$$

Note $\gamma_1 + \gamma_2 + \gamma_3 = (0,0,1)$ corresponds to an (anti) D0-brane.

The Chern character of an arbitrary instanton can be written as

$$\gamma = -(N_1\gamma_1 + N_2\gamma_2 + N_3\gamma_3), \qquad N_j \geq 0$$

For $X = \mathbb{P}^2$: $\kappa_{12} = \kappa_{23} = \kappa_{31} = 3$

The FI parameters ζ_i are chosen such $\phi_{31}^a = 0$ (Beilinson monad)
$\Rightarrow \zeta_1 \geq 0 \geq \zeta_3$
Moreover, quiver stability should agree with Gieseker stability of the sheaves for polarization J.

Then there is an equivalence

$$\mathcal{M}^Q(\vec{N}, \vec{\zeta}) \cong \mathcal{M}(\gamma; J),$$

with $\vec{\zeta}$ determined by $\vec{N}$ and J

Canonical stability

We call the polarization $J_0 = c_1(X)$ the "canonical polarization". This corresponds to the "canonical stability" for the quiver

$$\zeta_i^c = \rho\, \kappa_{ij}\, N^j + \eta_i$$

with $\rho \gg 1$. The first term on the rhs corresponds to the leading term for Gieseker stability, and the η_i to the subleading term.

Note that the first term differs by a sign from the attractor stability

The requirement on $\zeta_{1,3}^c$ implies for the first Chern class $\Rightarrow -N \leq c_1 \leq 0$, which can always be reached using tensoring by a line bundle, which is an isomorphism of moduli spaces.

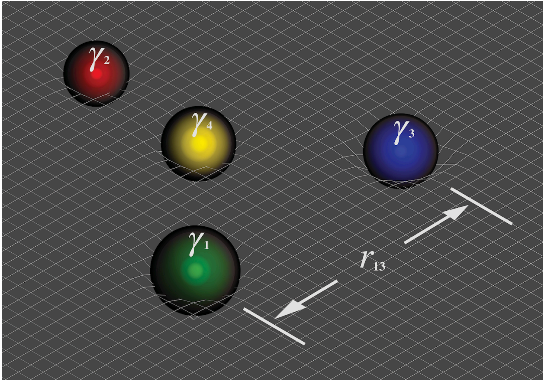
Question: Can we reproduce the invariants $\chi_y(\mathcal{M}(\gamma; J))$ from the quiver?

We have taken the approach of the attractor flow trees.

Denef (2001), Denef, Green, Raugas (2001), Denef (2002), Denef, Moore (2007), JM (2011), Alexandrov, Pioline (2018), Argüz, Bousseau (2021),...

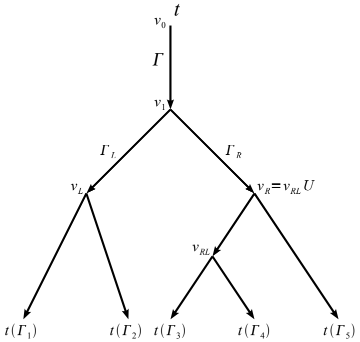
We use the "flow tree formula" for general number of centers by Alexandrov, Pioline (2018), proven by Argüz, Bousseau (2021). See the talks by Bousseau

Flow trees are originally developed for string theory compactifications on $\mathbb{R}^4 \times Y$, with Y a compact CY 3-fold. $\mathcal{N} = 2$ supergravity in $\mathbb{R}^4$ contains an intricate BPS spectrum of of black hole bound states.



Denef (2002,. . .), Denef, Moore (2007), JM, Pioline, Sen (2011,. . .),. . .

A flow tree is a 1-dimensional approximation to the 3-dimensional flow of the moduli (=stability conditions) under the attractor mechanism.



The (refined) index $\Omega(\gamma, y; t)$ can be expressed in terms of attractor indices $\Omega_*(\gamma)$. Thus the attractor points $t_*(\gamma)$ must exist in the space of stability conditions.

A vertex v represents a wall for Γ_{vL} and Γ_{vR}, thus the central charges satisfy

$$\mathrm{Im}(Z(\Gamma_{vL}, t_v)\,\bar{Z}(\Gamma_{vR}), t_v)) = 0$$

Furthermore, for a non-vanishing contribution, we have the following conditions at each vertex:

- $\kappa_{LR}\,\mathrm{Im}(Z(\Gamma_{vL}, t_{vU})\,\bar{Z}(\Gamma_{vR}, t_{vU})) > 0$
- $Z(\Gamma_{vL}, t_v)\,\bar{Z}(\Gamma_{vR}, t_v) > 0$

In favourable cases where the 2nd condition is automatic, the flow can be determined from node to node without calculating the explicit flow along the edges.

Result: All computed invariants for rank $|N| \geq 1$ sheaves on $\mathbb{P}^2$ are compatible with $\Omega_*(\vec{N}) = 0$ (for those $\vec{N}$ occuring in the flow trees), except $\Omega_*(\gamma_j) = 1$ for $j = 1, 2, 3$. The flow tree approach could be applied for $\sum_i N_i \leq 8$, which included moduli spaces for rank $N \leq 5$.

$[N; c_1; c_2]$	$\vec{N}$	$\Omega_c(\vec{N})$
$[1; 0; 2]$	$(1, 2, 2)$	$y^4 + 2y^2 + 3 + \dots$
$[1; 0; 3]$	$(2, 3, 3)$	$y^6 + 2y^4 + 5y^2 + 6 + \dots$
$[2; 0; 3]$	$(1, 3, 3)$	$-y^9 - 2y^7 - 4y^5 - 6y^3 - 6y - \dots$
$[2; -1; 2]$	$(1, 2, 1)$	$y^4 + 2y^2 + 3 + \dots$
$[2; -1; 3]$	$(2, 3, 2)$	$y^8 + 2y^6 + 6y^4 + 9y^2 + 12 + \dots$
$[3; -1; 3]$	$(1, 3, 2)$	$y^8 + 2y^6 + 5y^4 + 8y^2 + 10 + \dots$
$[4; -1; 4]$	$(1, 4, 3)$	$y^{14} + 2y^{12} + 5y^{10} + 10y^8 + 18y^6 + 28y^4 + 38y^2 + 42 + \dots$
$[4; -2; 4]$	$(1, 3, 1)$	$y^5 + y^3 + y + \dots$
$[4; -2; 5]$	$(2, 4, 2)$	$-y^{13} - 2y^{11} - 6y^9 - 10y^7 - 17y^5 - 21y^3 - 24y - \dots$

To understand this, we realized that the virtual dimension for these $\vec{N}$ at the attractor stability is negative:

$$\dim(\mathcal{M}^{Q}(\vec{N},\vec{\zeta}^{a})) < 0 \Longrightarrow \Omega_{*}(\vec{N}) = 0$$

Thus we could maybe say that this is a “genteel” spectrum?

Bridgeland (2016)

Torsion sheaves (with $N = 0$)

The D0-brane correspond to dimension vector $\vec{N} = (1,1,1) =: \delta$. It lies in the kernel of the anti-symmetric form $\langle \gamma, - \rangle$, and does not occur in the flow trees for $N \geq 1$ sheaves. Then,

$$\Omega_*(n\delta, y) = -y^{-1}(y^4 + y^2 + 1),$$

in agreement with the compact cohomology of $K_{\mathbb{P}^2}$

Mozgovoy, Pioline (2020), Mozgovoy (2021), Descombes (2021)

Thus we could say that the $\mathbb{P}^2$ quiver is "almost genteel"?

A similar structure was also found by Cordova, Neitzke (2013) for framed BPS states.

Other surfaces

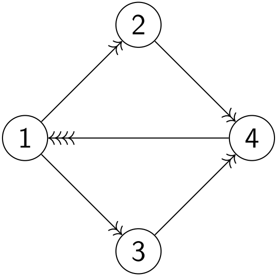
We did a similar analysis for other complex surfaces with an exceptional collection:

- Hirzebruch surfaces $\mathbb{F}_m$ (non-Fano for $m \geq 2$) $\Rightarrow$ different phases
- del Pezzo surfaces $dP_k = \mathbb{P}^2$ blown-up at a $k \leq 8$ points, toric for $k = 2, 3$ but non-toric for $k \leq 8$
- pseudo-del Pezzo surfaces PdP_k, $k \leq 6$

In agreement with calculations based on sheaves where available.

Hirzebruch surface $\mathbb{F}_0$ with base curve C and fiber F

Phase I:
Strong, full exceptional collection $\mathcal{C} = (\mathcal{O}, \mathcal{O}(C), \mathcal{O}(F), \mathcal{O}(C+F))$

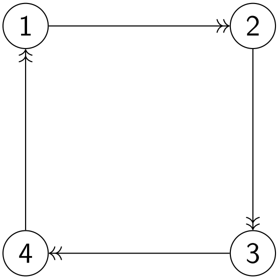


with cubic superpotential

$$
\begin{aligned}
W = {} & \phi^1_{12}\phi^1_{24}\phi^4_{41} - \phi^1_{12}\phi^2_{24}\phi^3_{41} - \phi^2_{12}\phi^1_{24}\phi^2_{41} + \phi^2_{12}\phi^2_{24}\phi^1_{41} \\
& - \phi^1_{13}\phi^1_{34}\phi^4_{41} + \phi^1_{13}\phi^2_{34}\phi^2_{41} + \phi^2_{13}\phi^1_{34}\phi^3_{41} - \phi^2_{13}\phi^2_{34}\phi^1_{41}
\end{aligned}
$$

Hirzebruch surface $\mathbb{F}_0$ with base curve C and fiber F

Phase II:
Strong, full exceptional collection
$\mathcal{C} = (\mathcal{O}, \mathcal{O}(C), \mathcal{O}(C+F), \mathcal{O}(2C+F))$



with quartic superpotential

$$W = \sum_{(\alpha\beta)\in S_2} \sum_{(\gamma\delta)\in S_2} \mathrm{sgn}(\alpha,\beta)\,\mathrm{sgn}(\gamma,\delta)\,\phi_{12}^{\alpha}\,\phi_{23}^{\gamma}\,\phi_{34}^{\beta}\,\phi_{41}^{\delta}\ . \qquad (1)$$

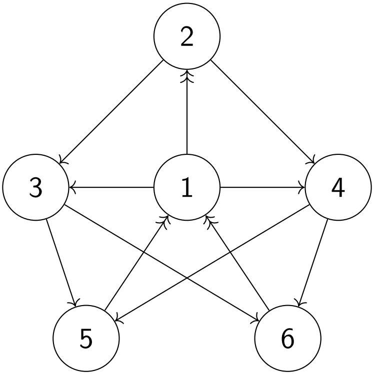
Hirzebruch surface $\mathbb{F}_0$ with base curve C and fiber F

Both quivers reproduce the results based on sheaves with rank $N > 0$ in their regime of validity, and with all attractor invariants vanishing except those corresponding to the nodes.
For D0-branes, the attactor indices are

$$\Omega_*(n\delta) = -y^{-1}(y^4 + b_2(X)y^2 + 1)$$

Mozgovoy, Pioline (2020), Descombes (2021)

More intricate quivers appear for higher del Pezzo's. For example for dP_3:



Conclusion

We have seen that for surfaces with a strong, full exceptional collection, attractor flow trees are an efficient and interesting technique to evaluate BPS indices.

The input of attractor indices $\Omega_*(\gamma)$ makes it possible to potentially apply the formalism more generally, such as other surfaces, CY 3-folds,...

Thank you!