Mock modular forms and instanton partition functions

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Modular forms

Definition: *A modular form of weight w is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$, which*

1. *transforms under an $SL_2(\mathbb{Z})$ matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as follows:*

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^w f(\tau),$$

2. *and whose growth for $\tau \to i\infty$ is such that*

$$\lim_{\tau \to i\infty} (c\tau + d)^{-w} f\left(\frac{a\tau + b}{c\tau + d}\right) \qquad (*)$$

is bounded for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$.

$\Longrightarrow$ f has a Fourier series:

$$f(\tau) = \sum_n a(n)\, q^n, \qquad q = e^{2\pi i \tau}$$

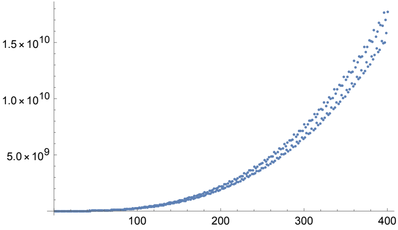
Example 1

Eisenstein series $k \in 2\mathbb{N}$, $k \geq 4$:

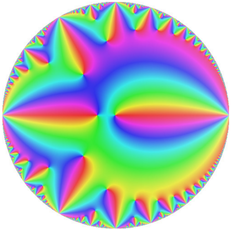
$$E_k(\tau) = 1 + \frac{(2\pi i)^k}{(k-1)!\,\zeta(k)} \sum_{n=1}^{\infty} \sigma_{k-1}(n)\, q^n$$

Divisor sum:

$$\sigma_k(n) = \sum_{d|n} d^k$$



Plot of coefficients of E_4



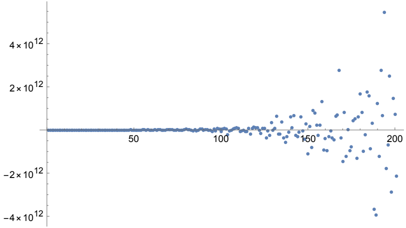
Tesselation of unit disc from E_4.

C. Feller, Summer Project TCD, 2018

Example 2

The Discriminant function $\Delta : \mathbb{H} \to \mathbb{C}$ is an example of a cusp form:

$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}$$
$$= \sum_{n=1}^{\infty} c(n)\, q^n$$



Ramanujan conjecture (proven by Deligne) for bound on $c(n)$:

$$|c(n)| \leq \sigma_0(n)\, n^{(12-1)/2}$$

Example 3

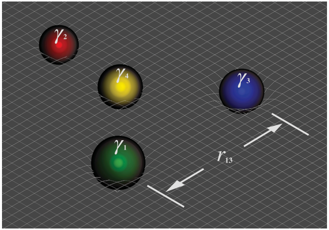
Example of a weakly holomorphic modular form:

$$\frac{1}{\Delta(\tau)} = \sum_{n=-1}^{\infty} d(n)\, q^n$$

Hardy-Rademacher-Ramanujan formula for growth of $d(n)$:

$$d(n) \sim e^{4\pi\sqrt{n}}$$

Crucial for the understanding of the microscopic entropy of CFT's and supersymmetric black holes Cardy;... Strominger, Vafa (1996);... ; Dabholkar (2004),...



Mock modular forms

Let

$$g^*(\tau,\bar{\tau}) = -2i \int_{-\bar{\tau}}^{i\infty} \frac{g(-v)}{(-i(v+\tau))^w}\,dv, \qquad (**)$$

Definition: *A mock modular form of weight w is a holomorphic q-series $f : \mathbb{H} \to \mathbb{C}$, such that its completion*

$$\widehat{f}(\tau,\bar{\tau}) = f(\tau) + g^*(\tau,\bar{\tau})$$

1. *transforms as a modular form of weight w,*
2. *g^* is the image under the map $(**)$ of the complex conjugate g of a modular form with weight $2-w$,*
3. *The combination $(*)$ is bounded for f.*

Release poster for 2015 movie portraying Ramanujan's life.

Ramanujan; Zwegers (2008); Zagier (2010);...

Example 4

Generating function of Hurwitz class numbers $H(n)$,

$$h_\mu(\tau) = \sum_{n\geq 0} H(4n-\mu)\, q^{n-\mu/4}, \qquad \mu = 0, 1$$

These functions are mock modular forms of weight 3/2 for the congruence subgroup $\Gamma^0(4)$, and with

$$g_0(\bar\tau) = \overline{\vartheta_2(2\tau)}, \qquad g_1(\bar\tau) = \overline{\vartheta_3(2\tau)}$$

Zagier (1975)

Example 5

Mock modular form of weight 1/2 for $SL_2(\mathbb{Z})$ central to the Mathieu Moonshine phenomenon:

$$F(\tau) = q^{-1/8}(-2 + 90q + 462q^2 + 1540q^3 + \dots)$$

with

$$g(\bar{\tau}) = \overline{\eta(\tau)}^3$$

Note this function is weakly holomorphic.

Eguchi, Hikami (2009), Dabholkar, Murthy, Zagier (2012),...

Four-manifolds and lattices

Let X be an oriented, smooth, compact four-manifold. We assume X is simply connected (thus $b_1 = \dim(H^1(X)) = 0$).

$H^2(X, \mathbb{Z})$ together with the intersection form

$$B(\boldsymbol{k}_1, \boldsymbol{k}_2) = \int_X \boldsymbol{k}_1 \wedge \boldsymbol{k}_2, \qquad \boldsymbol{k}_{1,2} \in H^2(X, \mathbb{Z})$$

gives rise to an integral, uni-modular lattice L (the image of $H^2(X, \mathbb{Z})$ in $H^2(X, \mathbb{R})$)

The lattice has signature (b_2^+, b_2^-).

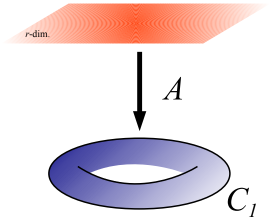
For $b_2^+ = 1$, let J be the normalized generator of the unique self-dual direction in $H^2(X, \mathbb{R})$. It provides the projection of $\boldsymbol{k} \in L$ to $(L \otimes \mathbb{R})^+$,

$$\boldsymbol{k}_+ = B(\boldsymbol{k}, J)\, J$$

Instantons solutions in Yang-Mills theory

Let $P \to X$ be a G-principal bundle for a Lie group G.

Let $A = A_\mu dx^\mu \in \Omega^1(X, \mathrm{ad}(P))$ be the connection 1-form with associated field strength $F = dA + A \wedge A$.



Instantons solutions in Yang-Mills theory

Instantons are the anti-self-dual solutions

$$F = - * F$$

They are characterized by topological numbers:

Instanton number:

$$k = \frac{1}{8\pi^2} \int_X \mathrm{Tr}[F \wedge F] \in \mathbb{Z}$$

't Hooft flux:

$$c_1 = 2\boldsymbol{\mu} = \frac{i}{2\pi} \mathrm{Tr}[F] \in H^2(X, \mathbb{Z})$$

Moduli spaces

The moduli space of instantons is defined as:

$$\mathcal{M}_{k,\boldsymbol{\mu}} = \{\, F = -*F \,|\, \text{modulo gauge transformations}\}$$

Virtual dimension for $G = SU(2)$:

$$\mathrm{vdim}_{\mathbb{R}}(\mathcal{M}_{k,\boldsymbol{\mu}}) = 8k - 3(b_2^+ + 1)$$

$\mathcal{M}_{k,\boldsymbol{\mu}}$ may be non-compact or have singularities

The spaces $\mathcal{M}_{k,\boldsymbol{\mu}}$ have been crucial for the classification of four-manifolds up to diffeomorphism.

Rigorous mathematical results are available through the correspondence of the instantons and semi-stable vector bundles by Hitchin-Kobayashi and Donaldson-Uhlenbeck-Yau

Instanton partition functions are generating series

$$Z_{\boldsymbol{\mu}}(Y) = \sum_k c_{\boldsymbol{\mu}}(k)\, Y^k$$

where the $c_{\boldsymbol{\mu}}(k)$ is a topological invariant of $\mathcal{M}_{\boldsymbol{\mu},k}$ and Y a formal variable for the moment

Examples:

- Donaldson invariant:

$$D_{\ell,n}(p,\boldsymbol{x}) = \int_{\mathcal{M}_{k,\boldsymbol{\mu}}} \mu_D(p)^\ell \wedge \mu_D(\boldsymbol{x})^n$$

  with the Donaldson map $\mu_D : H_r(X) \to H^{4-r}(\mathcal{M}_{k,\boldsymbol{\mu}})$
- Euler characteristic: $\chi(\mathcal{M}_{\boldsymbol{\mu},k}) = \sum_j (-1)^j\, b_j(\mathcal{M}_{\boldsymbol{\mu},k})$
- ...

Yang-Mills action:

$$S[A] = -\frac{1}{g^2}\int_X \mathrm{Tr}[F \wedge *F] + \frac{i\theta}{8\pi^2}\mathrm{Tr}[F \wedge F]$$

Complexified coupling:

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} \in \mathbb{H}$$

S-duality is an equivalence of the theory under

- T: $\tau \to \tau + 1$
- S: $\tau \to -1/\tau$, $G \to {}^L G$

Exact non-perturbative results are often difficult to achieve in quantum field theory. Supersymmetric theories contain more fields, which however reduce quantum effects, and allow for exact results.

Path integrals localize on Q-fixed equations of topologically twisted $\mathcal{N}=2$ and $\mathcal{N}=4$ supersymmetric theories. Instantons solve the Q-fixed equations.

The physical approach provides a useful viewpoint on the mathematical invariants mentioned before. Together with S-duality, this provides an heuristic explanation for the role of modularity for instanton partition functions.

The analysis for $\mathcal{N}=2$ and $\mathcal{N}=4$ are different, but mock modular forms play a key role for both.

$\mathcal{N} = 2$ supersymmetry: u-plane integral

The physical path integral leads to the finite dimensional u-plane integral for four-manifolds with $b_2^+ = 1$.

Witten (1995); Moore, Witten (1997); Losev, Nekrasov, Shatashvili (1997), ...

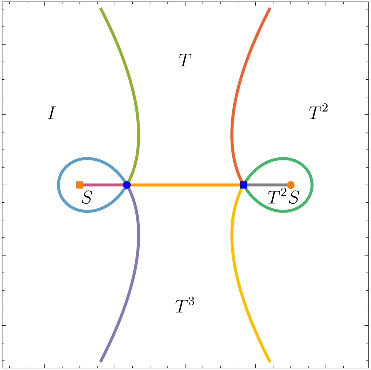
For the simplest case, pure $\mathcal{N} = 2$ supersymmetric $SU(2)$ YM, u is the Hauptmodul for the congruence subgroup $\Gamma^0(4) \in SL_2(\mathbb{Z})$,

$$\begin{aligned}\frac{u(\tau)}{\Lambda^2} &= \frac{1}{2}\frac{\vartheta_2(\tau)^4 + \vartheta_3(\tau)^4}{\vartheta_2(\tau)^2\vartheta_3(\tau)^2} \\ &= \frac{1}{8}(q^{-1/4} + 20q^{1/4} - 62q^{3/4} + 216q^{5/4} + \mathcal{O}(q^{7/4})),\end{aligned}$$

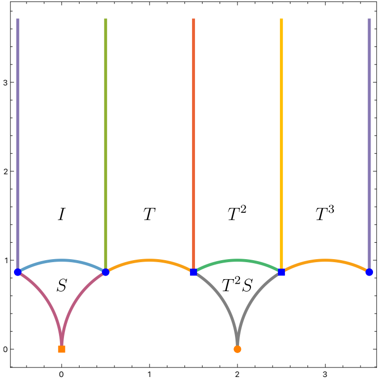
with ϑ_j Jacobi theta series and Λ the scale of the quantum field theory.

Seiberg, Witten (1994); Matone (1996); Nahm (1996),...

u-plane versus $\mathbb{H}/\Gamma^0(4)$



Partitioning of the u-plane by the images of $\mathcal{F}_I$ in $\mathbb{H}/\Gamma^0(4)$



Fundamental domain for $\mathbb{H}/\Gamma^0(4)$

The u-plane integral takes the form

$$\Phi^J_{\boldsymbol{\mu}}[\mathcal{O}] = \int_{\mathbb{H}/\Gamma^0(4)} d\tau \wedge d\bar{\tau}\, \nu(\tau)\, \mathcal{O}\, \Psi^J_{\boldsymbol{\mu}}(\tau,\bar{\tau})$$

with

- $\nu(\tau) = \frac{da}{d\tau} A(u)^{\chi(X)} B(u)^{\sigma(X)} = q^{-\frac{3}{8}} + \dots$
- Sum over fluxes:

$$\Psi^J_{\boldsymbol{\mu}}(\tau,\bar{\tau}) = \frac{1}{\sqrt{y}} \sum_{\boldsymbol{k}\in L+\boldsymbol{\mu}} B(\boldsymbol{k},J)\, q^{-\boldsymbol{k}_-^2/2}\, \bar{q}^{\boldsymbol{k}_+^2/2}$$

with $y = \mathrm{Im}(\tau)$

Moore, Witten (1997)

Efficient evaluation using mock modular forms for all X with $b_2^+ = 1$

Malmendier (2011); Malmendier, Ono (2012); Korpas, JM (2017); Korpas, JM, Moore, Nidaiev (2019); JM, Moore (2021)

Construction of a suitable anti-derivative:

$$\frac{\partial \widehat{F}(\tau,\bar\tau)}{\partial \bar\tau} = \Psi_{\boldsymbol{\mu}}^J(\tau,\bar\tau),$$

Then

$$\Phi_{\boldsymbol{\mu}}^J[\mathcal{O}] = [\mathcal{O}\,\nu(\tau)\,F(\tau)]_{q^0} + \text{contributions from other cusps},$$

with $F(\tau) = \sum_n c(n)\,q^n$ the holomorphic part of $\widehat{F}(\tau,\bar\tau)$

Explicit results for $X = \mathbb{P}^2$

Choose for X the complex projective plane $\mathbb{P}^2$

Let $\mathcal{O} = u^\ell \simeq \mu_D(\ell\, p)$

Then

ℓ	$8^\ell\, \Phi^J_{1/2}[u^\ell]$
0	1
2	19
4	680
6	29 557
8	1 414 696
$\vdots$	$\vdots$

In agreement with results from Ellingsrud, Göttsche (1995) and Göttsche, Zagier (1996) for $\mathbb{P}^2$

Large charge correlators

With these modular expressions, we can study the asymptotics for large ℓ, or "large charge" correlators

We find experimental evidence that

- $\Phi^J_{\boldsymbol{\mu}}[u^\ell] \sim \frac{C}{\ell \log(\ell)}$
- $\Rightarrow$ Evidence that $\Phi^J[e^{2pu}]$ is an entire function of p

Other observables and physical theories

Surface operators appear as as (holomorphic) elliptic variables in Ψ_{μ}^{J}

Background fluxes can be included for each $U(1)$ subgroup of the global symmetry group. These couplings lead to non-holomorphic elliptic variables.

Gives rise to:

- an elliptic refinement of $F(\tau)$:

$$F(\tau,z) = q^{-1/8}(-w^{1/2} - w^{-1/2} + (-16w^2 + 19w^{3/2} - 64w + 26w^{1/2} + 160 + \text{palindromic terms})q + \dots)$$

  with $w = e^{2\pi i z}$. Do these coefficients have a Moonshine interpretation? Aspman, Furrer, JM (2023)
- many explicit results with similar "large charge" asymptotics questions as above
- evaluation of the u-plane integral with elliptic variables is subtle

$\mathcal{N} = 4$ supersymmetry (Vafa-Witten theory)

The partition function for $G = SU(N)$ of this theory takes the form

$$Z_N^X(\tau) = \sum_{k \geq 0} b_N(k)\, q^{k - N\chi(X)/24}$$

with

$$b_N(k) = \chi(\mathcal{M}_{k,\boldsymbol{\mu}})$$

Physical expectation: Z_N^X transforms as a weakly holomorphic modular form of weight $-\chi(X)/2$

Vafa, Witten (1994)

NB $\mathcal{M}_{k,\boldsymbol{\mu}}$ includes pointlike, singular instantons. Let $\mathcal{N}_{k,\boldsymbol{\mu}}$ be the moduli space of smooth instantons

As a result of the relation between these moduli spaces, Z_N^X can be expressed for an algebraic surface X as

$$Z_N^X(\tau) = \frac{f_N^X(\tau)}{\eta(\tau)^{N\chi(X)}}$$

with η the Dedekind eta function and

$$f_N^X(\tau) = \sum_{k\geq 0} c_N(k)\, q^k$$

and

$$c_N(k) = \chi(\mathcal{N}_{k,\boldsymbol{\mu}})$$

Göttsche (1998)

Motivates the analysis of both $b_N(k)$ and $c_N(k)$

$G = SU(2)$ and $X = \mathbb{P}^2$

Asymptotics of b_N follow from weak holomorphicity of Z_N^X:

$$b_N(k) \sim e^{\pi\sqrt{2kN\chi(X)}/3}$$

Generating functions f_N are determined for arbitrary N in terms of (generalized) Appell functions using algebraic-geometric techniques.

Klyachko (1991), Yoshioka (1994), Vafa, Witten (1994), Kool (2010), Weist (2010), JM (2010), (2014), (2017)

Exact Rademacher type formula for b_N, $N = 2, 3$ and $X = \mathbb{P}^2$

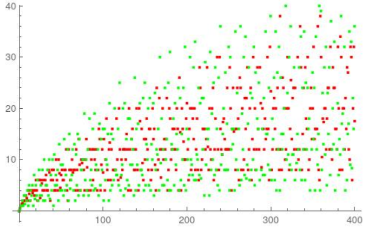
Bringmann, JM (2013), Bringmann, Nazaroglu (2019)

For $SU(2)$:

$$f_{2,\mu}(\tau) = 3h_\mu(\tau)$$

with h_0 the generating function of Hurwitz class numbers
$\Rightarrow$ f_2 is a mock modular form

Plot of coefficients for $N = 2$



Plot of the coefficients $H(4n - \mu)$ as function of n. The red dots represent the coefficients $\mu = 0$, while the green ones represent the coefficients $\mu = 1$.

The distribution of the coefficients appears chaotic and highly scattered. On average the coefficients appear to grow as a power law, in agreement with the growth for an Eisenstein series.

Thus almost all the cohomology of $\mathcal{M}_{k,\boldsymbol{\mu}}$ is due to the point-like instantons

JM, Chattopadhyaya (2023)

$N = 3$

The first few terms of the q-expansions of $f_{3,\mu}(\tau)$ are:

$$\begin{aligned} f_{3,0}(\tau) &= \frac{1}{9} - q + 3q^2 + \ldots \\ f_{3,1}(\tau) &= f_{3,2}(\tau) = 3q^{5/3} + 15q^{8/3} + 36q^{11/3} + \ldots \end{aligned}$$

$f_{3,\mu}$ are examples of a mock modular form of depth 2 and weight 3.

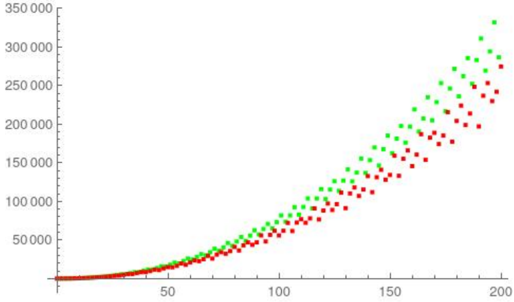
The modular completion reads

$$\widehat{f}_{3,\mu} = f_{3,\mu}(\tau) - \frac{i}{\pi}\left(\frac{3}{2}\right) \sum_{\nu=0,1} \int_{-\bar{\tau}}^{i\infty} \frac{\widehat{f}_{2,\nu}(\tau,-v)\,\Theta_{\mu/2}(3v)}{(-i(v+\tau))^{3/2}}\, dv,$$

which involves an iterated integral

Manschot (2019). See also Alexandrov, Banerjee, JM, Pioline (2018); Bringmann, Kaszian, Milas (2019).

We plot the coefficients $b_{3,0}(n)$ of $f_{3,0}$



The green dots represent the coefficients for odd n and the red dots for even n

We observe a fairly regular power law growth $\sim n^2$, in agreement with that of Eisenstein series of weight 3.

Mock cusp form

Definition: *A mock cusp form is a mock modular form $f(\tau)$ such that the combination $(*)$ vanishes for all elements of $SL_2(\mathbb{Z})$. Thus in particular the constant term of the Fourier series of f vanishes.*

The weight 3 Eisenstein series m_μ,

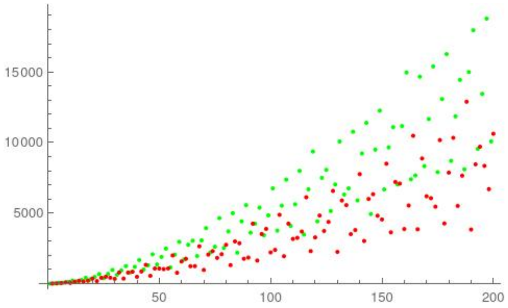
$$\begin{aligned} m_0(\tau) &= \frac{1}{9} + 8q + 30q^2 + \dots \\ m_1(\tau) &= 3q^{2/3} + 24q^{5/3} + 51q^{8/3} + \dots \end{aligned}$$

transforms identically under $SL_2(\mathbb{Z})$ as $\widehat{f}_{3,\mu}$

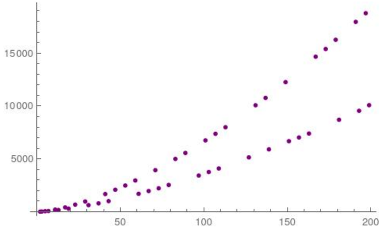
Their difference is the mock cusp form S_μ:

$$\begin{aligned} S_0(\tau) &= \sum_n s_0(n)q^n = 3q + 9q^2 + 21q^3 + \dots, \\ S_1(\tau) &= \sum_n s_1(n)q^n = q^{2/3} + 7q^{5/3} + \dots. \end{aligned}$$

Plot of coefficients $s_0(n)$ of S_0



The green dots represent the coefficients $s_0(n)$ for odd n and the red dots for even n



Plot of prime coefficients $s_0(p)$ of S_0

The growth of the coefficients is surprisingly regular. The least square fits for the primes p are

$$s_0(p) \sim \begin{cases} 6.7547\, p^{3/2}, & p = 3n+1, \\ 3.578\, p^{3/2}, & p = 3n-1. \end{cases}$$

$$s_1(p/3) \sim 0.9935 p^{3/2}$$

Based on this, we conjecture that the growth is $\sim n^{3/2}$.
This is intermediate between the growth of weight 3 Eisenstein series ($\sim n^2$) and cusp forms $\sim n$.

Basic saddle point method

$$S_0(\tau) = \frac{-i\tau^3}{\sqrt{3}}\left(S_0(-1/\tau) + 2S_1(-1/\tau) + \mathcal{J}_0(-1/\tau)\right)$$

$$\begin{aligned}\mathcal{J}_0(\tau) &= \frac{3\sqrt{3}i}{2\sqrt{2}\pi}\sum_{\nu=0}^{2}\sum_{\alpha=0}^{1}\int_0^{i\infty}\frac{\hat{f}_{2,\alpha}(\tau,w)\Theta_{\frac{2\nu+3\alpha}{6}}(3w)}{(-i(w+\tau))^{3/2}}\,dw \\ &= -\frac{\sqrt{3}i}{2\sqrt{2}\pi}\frac{1}{4}\frac{2}{\sqrt{-i\tau}} + O(\tau^{-1}).\end{aligned}$$

Then

$$|s_\mu(n)| \leq \int_0^1 \left|S_\mu(\tau)\, e^{-2\pi i n\tau}\right| d\tau < C\, n^{5/2}$$

for some constant C

Weaker but consistent with the conjectured $\sim n^{3/2}$

Chattopadhyaya, JM (2023)

Some questions:

- Can a more accurate estimate be obtained?
- What about the growth for $N = 2, 3, \ldots$?
- Is there a geometric or physical understanding for these growth patterns?

Concluding comments

- Mock modular forms play a key role for instanton partition functions
- These partition functions give rise to new modular-type functions
- The physics and geometry of instantons motivate the analysis of their transformation properties and Fourier coefficients

Thank you!