**Linear algebra I**
**Sample exam solutions**

**1a.** One can compute the inverse of the given matrix using the row reduction

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 4 & 0 & 1 & 0 \\ 1 & 3 & 9 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -3 & 1 \\ 0 & 1 & 0 & -5/2 & 4 & -3/2 \\ 0 & 0 & 1 & 1/2 & -1 & 1/2 \end{array}\right].$$

**1b.** Let $f(x) = a + bx + cx^2$ be the desired polynomial. Then we must have

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 10 \end{bmatrix},$$

where the leftmost matrix is the given one. Multiplying by its inverse, we find

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 & -3 & 1 \\ -5/2 & 4 & -3/2 \\ 1/2 & -1 & 1/2 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 10 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}.$$

**2a.** Suppose $A, B$ are $n \times n$ lower triangular matrices and $i < j$, in which case

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

If $i < k$, then $a_{ik} = 0$ and the summand is zero. Otherwise, $k \leq i < j$ so $b_{kj} = 0$ and the summand is still zero. This means that $(AB)_{ij} = 0$, so $AB$ is lower triangular.

**2b.** Suppose $A$ is $n \times n$ lower triangular. Using row reduction, one finds that

$$\det A = a_{11} \det \begin{bmatrix} 1 & & & \\ a_{21} & a_{22} & & \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = a_{11} \det \begin{bmatrix} 1 & & & \\ 0 & a_{22} & & \\ \vdots & \vdots & \ddots & \\ 0 & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

We now proceed in the same manner, factoring out $a_{22}$ and clearing all entries below the pivot. Applying this approach repeatedly, we end up with $\det A = a_{11} a_{22} \cdots a_{nn}$.

**3a.** Suppose that $y$ is in the column space of $B$. Then we have

$$\begin{aligned} y = Bx \text{ for some } x \in \mathbb{R}^n \quad &\Longrightarrow \quad Ay = A(Bx) \text{ for some } x \in \mathbb{R}^n \\ &\Longrightarrow \quad Ay = (AB)x = 0 \text{ for some } x \in \mathbb{R}^n \\ &\Longrightarrow \quad y \text{ is in the null space of } A. \end{aligned}$$

**3b.** The reduced row echelon form of $A$ is

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 11 \\ 0 & 1 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & -5 \end{bmatrix}.$$

Thus, the null space and the column space of $A$ are

$$\mathcal{N}(A) = \operatorname{Span}\left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -11 \\ 3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}, \qquad \mathcal{C}(A) = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

**4a.** Yes, it does follow. Suppose some linear combination of the $\boldsymbol{v}_i$'s is zero, say

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \ldots + c_n\boldsymbol{v}_n = 0.$$

Since $T(0) = 0$ by linearity, we must then have

$$c_1T(\boldsymbol{v}_1) + c_2T(\boldsymbol{v}_2) + \ldots + c_nT(\boldsymbol{v}_n) = T(c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \ldots + c_n\boldsymbol{v}_n) = 0.$$

As the vectors $T(\boldsymbol{v}_i)$ are linearly independent, this implies that $c_i = 0$ for all $i$.

**4b.** Since $T(\boldsymbol{e}_1) = \boldsymbol{e}_2$ and $T(\boldsymbol{e}_2) = \boldsymbol{e}_1$, the matrix with respect to the standard basis is

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \Longrightarrow \quad T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x\boldsymbol{e}_2 + y\boldsymbol{e}_1 = \begin{bmatrix} y \\ x \end{bmatrix}.$$

To find the matrix with respect to the basis $B$, we compute the images

$$T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad T\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

and express those in terms of the elements of $B$. Using the row reduction

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 3 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 5 & 8 \\ 0 & 1 & -3 & -5 \end{bmatrix},$$

we conclude that the desired matrix is

$$J = \begin{bmatrix} 5 & 8 \\ -3 & -5 \end{bmatrix}.$$