Class Polynomials for Non-holomorphic Modular Functions

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The Modular Invariant and its Special Values

- The *j-function* is an important example of a modular function

\[
j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + \ldots \quad (q := e^{2\pi i \tau}).
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The Modular Invariant and its Special Values

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The Modular Invariant and its Special Values

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- \textbf{Singular moduli} are values of the \textit{j}-invariant at quadratic irrationalities.

- Here are several examples:
  \[ j(i) = 1728, \quad j \left( \frac{1 + i\sqrt{7}}{2} \right) = -3375, \quad j(i\sqrt{2}) = 8000. \]
Singular moduli generate “class fields”.

Strange consequence:

$e^{\pi \sqrt{163}} = 262537412640768743999999999925 \in \mathbb{Z} + \epsilon^2$. 
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Hilbert Class Polynomials

**Definition**

The *class polynomial* of discriminant $D$ is:

\[ H_D(x) := \prod_{1 \leq i \leq h(D)} (x - j(\tau_D, i)) \in \mathbb{Z}[x]. \]
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Theorem
For all $D$, $H_D(x)$ is irreducible in $\mathbb{Z}[x]$ and its splitting field is a class field.
Computing Hilbert Class Polynomials

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Zagier’s seminal paper *Traces of Singular Moduli* gives an automatic procedure for computing class polynomials.
Zagier Grids

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$$f_0 = 1 + 2q + 2q^4 + 2q^9 + \ldots$$

$$f_3 = q^{-3} - 248q + 26752q^4 - 85995q^5 + \ldots$$
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\]
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$$H_D(j(\tau)) = q^{-H(d)} \prod_{n=1}^{\infty} (1 - q^n)^{A(n^2, d)}.$$
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**Remark**

1. Zagier’s theory provides a new proof of Borcherds’ theorem and he shows that \( A(1,d) \) is the trace of singular moduli.
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**Remark**

1. Zagier’s theory provides a new proof of Borcherds’ theorem and he shows that \( A(1,d) \) is the trace of singular moduli.
2. Zagier’s work applies to a much more general class of forms.
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Example: These appear in recent work of Bruinier-Ono on $p(n)$. 
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Traces of Singular Moduli

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**Definition**

Let $Q_d$ be the set of positive definite binary quadratic forms of discriminant $d$. For a modular function $F$, define the trace:

$$\text{Tr}_d(F) := \sum_{Q \in Q_d/\Gamma} w_Q^{-1} F(\tau_Q).$$
An Example of Zagier’s Theory

**Theorem (Zagier)**

Let

\[ J(z) := j(z) - 744 \]

and

\[ g(z) := \theta_1(z) \frac{E_4(4z)}{\eta(4z)^6} = \sum B(d)q^n \]

For any positive integer \( d \equiv 0, 3 \pmod{4} \), we have
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\[ \text{Tr}_{-d}(J(z)) = -B(d). \]
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Remark: It appears that the third symmetric function is always an integer.
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Define $H_d(K; x) := \prod_{Q \in Q_d/\Gamma} (x - K(\tau_Q))$.

- $H_{-23}(K; x) = x^3 - 23261998x^2 - \frac{3945271661}{23}x - 7693330369871$. 
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It appears that the third symmetric function is always an integer.
A Natural Question

Theorem

The fields generated by these singular moduli are contained in the “correct” class fields.
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Question (Zagier ?)

What is the obstruction to integrality of these coefficients, and what is the pattern of their denominators?
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**Theorem**

*The fields generated by these singular moduli are contained in the “correct” class fields.*

**Question (Zagier ?)**

*What is the obstruction to integrality of these coefficients, and what is the pattern of their denominators?*

**Answer**

*Our theorem predicts the correct/sharp denominators.*
The Maass raising operator, raises the weight by 2:
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Traces for Negative Weight Forms

- The **Maass raising operator**, raises the weight by 2:
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- For \( f \) of negative weight, \( \partial f \) is the iterated raising to weight 0.
Our Main Result

Theorem (G-R)

Let \( f(z) \in M^!_k, 0 > k \in 2\mathbb{Z} \) have integral principal part. Denote the \( n^{th} \) symmetric function in the singular moduli of discriminant \( d \) for \( \partial f \) by \( S_f(n; d) \). Let

\[
B(n, k) := \begin{cases} 
\frac{-nk}{4} & \text{if } nk \in 4\mathbb{Z} \\
\frac{1}{4}(-nk + 2k - 2) & \text{otherwise.}
\end{cases}
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Our Main Result

**Theorem (G-R)**

Let $f(z) \in M^!_k$, $0 > k \in 2\mathbb{Z}$ have integral principal part. Denote the $n^{th}$ symmetric function in the singular moduli of discriminant $d$ for $\partial f$ by $S_f(n; d)$. Let

$$B(n, k) := \begin{cases} \frac{-nk}{4} & \text{if } nk \in 4\mathbb{Z} \\ \frac{1}{4}(-nk + 2k - 2) & \text{otherwise.} \end{cases}$$

Then if $(p, d) = 1$, we have that $S_f(n; d)$ is $p$-integral. If $p|d$ is good for $(k, N)$, we have that

$$p^{B(n, k)} \cdot S_f(n; d) \text{ is } p\text{-integral.}$$
Corollary

For any $f(z) \in M^{!}_{-2}$ with integral principal part, we have that

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Remark

This theorem is sharp.
Sketch of Proof

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- We prove the following fact.
The Spectral Decomposition

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Let $F$ be a product of “raises” of modular forms. Then there are modular forms $g_j \in M^!_{k-2j}$ such that
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$$F = \sum_{j=0}^{E} R^j g_j,$$

Remark

The proof gives an explicit algorithm for computing the forms $g_j$. 
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Sketch of Proof (cont).

- Work of Duke and Jenkins allows us to study integrality of traces for $\partial f$ when $f$ is a negative weight modular form.
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- Bounding denominators on each piece gives a naïve bound.

- However, this falls far short of our theorem.
Two Intervening Problems

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  We prove a vanishing condition on which forms in the decomposition actually appear.

- **Obstruction 2**: The coefficients $c_{i,j}$ in the previous theorem also introduce artificial denominators.

  We show that they cancel using the action of the Hecke algebra on Poincaré series.

  Q.E.D.
Proof of the Spectral Decomposition

- Using the iterated lowering operator $L^n$, for large $n$ this kills $F$. 
Proof of the Spectral Decomposition

- Using the iterated lowering operator $L^n$, for large $n$ this kills $F$.
- Using the intertwining properties of the Maass lowering and raising operators, we get the recursion:

\[
\begin{align*}
\text{g}_E &= L_E F_c, \\
\text{g}_i &= 1_c, \\
L_i F - E \sum_{j = i+1} \text{c}_{i,j} R_j - i \text{g}_j.
\end{align*}
\]

Here $c_{i,j} := j != (-k+j+i)! (-k+j)!$. 
Proof of the Spectral Decomposition

- Using the iterated lowering operator $L^n$, for large $n$ this kills $F$.
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$$g_E = \frac{L^E F}{c_{E,E}},$$

where $c_{i,j} = j!(-k+j+i)!/(j-i)!(-k+j)!$. 
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- Using the intertwining properties of the Maass lowering and raising operators, we get the recursion:

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$$g_i = \frac{1}{c_{i,i}} \left( L^i F - \sum_{j=i+1}^{E} c_{i,j} R^{j-i} g_j \right).$$
Proof of the Spectral Decomposition

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- Here

$$c_{i,j} := \frac{j!(-k + j + i)!}{(j - i)!( -k + j)!}.$$

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Let
\[
\text{Tr}^*_{d,D}(f) := (-1)^{\left\lfloor \frac{s-1}{2} \right\rfloor} \left| d \right|^{-\frac{s}{2}} \left| D \right|^{\frac{s-1}{2}} \text{Tr}_{d,D}(\partial f).
\]
Let

$$\text{Tr}^{*}_{d,D}(f) := (-1)^{\left\lfloor \frac{\hat{s} - 1}{2} \right\rfloor} |d|^{-\frac{s}{2}} |D|^{-\frac{s-1}{2}} \text{Tr}_{d,D}(\partial f).$$

They define the $D^{th}$ Zagier lift of $f$:

$$Z_{D}(f) := \sum_{m \geq 0} b(m)q^{-m} + \sum_{dD < 0} \text{Tr}^{*}_{d,D}(f)q^{|d|}.$$
Duke and Jenkins’ Theorem

Theorem (Duke-Jenkins)

Suppose that $f \in M_k^1$, $k \leq 0$. If $f \in \mathbb{Z}[[q]]$, then $\mathcal{Z}(f)$ is a half-integral weight modular form with integral coefficients.
A Useful Vanishing Criterion

**Definition**

Let $0 > k \in 2\mathbb{Z}$ and $n \in \mathbb{N}$. We say $m$ is a **bad weight** for $(k, n)$ if $m$ is of the form $kn + 4i + 2$ for $0 \leq i \leq -\frac{k}{2} - 1$. 
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Let $0 > k \in 2\mathbb{Z}$ and $n \in \mathbb{N}$. We say $m$ is a bad weight for $(k, n)$ if $m$ is of the form $kn + 4i + 2$ for $0 \leq i \leq -\frac{k}{2} - 1$.

Theorem (G-R)

Let $f \in M_k^!$ and consider the product $F = (\partial f)^n$. Decompose $F = \sum \partial(g_i)$. Then if $g_i$ has bad weight for $(k, n)$, $g_i \equiv 0$. 
Rankin-Cohen Brackets

Let $f \in M_k^!, g \in M_\ell^!, n \in \mathbb{N}$. The $n^{th}$ Rankin-Cohen bracket is
Let $f \in M^!_k$, $g \in M^!_\ell$, $n \in \mathbb{N}$. The $n^{th}$ Rankin-Cohen bracket is

$$[f, g]_{n}^{(k, \ell)} := \sum_{r+s=n} (-1)^{r} \binom{n + k - 1}{s} \binom{n + \ell - 1}{r} f(r) \cdot g(s).$$
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$$[f, g]_n^{(k, \ell)} := \sum_{r+s=n} (-1)^r \binom{n+k-1}{s} \binom{n+\ell-1}{r} f(r) \cdot g(s).$$

This gives an (essentially unique) map

$$[\cdot, \cdot]_n^{(k, \ell)} : M_k^! \otimes M_\ell^! \to M_{k+\ell+2n}^!.$$
Products of Two Forms

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- We can expand in terms of Rankin-Cohen brackets.

\[ \sum_{m=0}^{j} (-1)^{j+m} \cdot (m+r)^{s} \cdot (m-r-1)^{s} = 0. \]
Products of Two Forms

- We need a **vanishing** condition for the product of two forms.

- We can expand in terms of Rankin-Cohen brackets.

- Using a calculation of Beyerl-James-Trentacoste-Xue, this reduces to a binomial sum identity, for $j$ odd

$$
\sum_{m=0}^{s} (-1)^{(j+m)} \cdot \frac{(m+r)}{j} \cdot \frac{(s)}{m} \cdot \frac{(m-r-1)}{r+m-j} \cdot \frac{(-r-2s+m+j-1)}{m+r-j} = 0.
$$
Obstruction 2: Lining Up Principal Parts

- Raise the Zagier lifts of the pieces to the same weight and let:

\[
Z(\tau) := \sum_{t=0}^{\left\lfloor \frac{E+1}{2} \right\rfloor} (-1)^{M+t} R^{M+t} \zeta_{1}(g_{2t-1}) + \sum_{t=0}^{M} (-1)^{M+t} R^{M-t} \zeta_{1}(g_{2t}).
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- By comparison with \( F \), we observe that the holomorphic part \( Z^+ \) of \( Z \) has integral principal part.

- If all the coefficients of \( Z^+ \) are integral, then the \( c_{i,j} \)-denominators will cancel.
Maass-Poincaré Series

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- Thus, for any $F(\tau) = \sum a(n)q^n \in M_{-2k}^!$ we can write

$$F = \sum_{n<0} a(n)n^{1+2k} f_{-2k,1} \mid T(n).$$
Maass-Poincaré Series

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  \[
  F = \sum_{n<0} a(n)n^{1+2k}f_{-2k,1}|T(n).
  \]

- The Zagier lift is equivariant with the Hecke action:
  \[
  \mathcal{Z}_D(f|T(n)) = \mathcal{Z}_D(f)|T(n^2).
  \]
Hypotheses

- For the next few slides, we suppose $k$ and $n$ are positive integers with $k$ even.
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- We assume $p$ is ordinary for all eigenforms in a basis of $S_k$. 
$p$-adic Properties

**Theorem (G-R)**

*Then is a Hecke operator $\mathcal{H}_n$ such that*

\[
\left( f^k, 1 \middle| \mathcal{H}_n \right) \mid_{\mathcal{H}_n} \equiv f^k, 1 \mid_{\mathcal{H}_n} \pmod{p^m}.
\]
Theorem (G-R)

Then is a Hecke operator $\mathcal{H}_n$ such that

$$f_{2-k,1} | \mathcal{H}_n \in M_{2-k}^!,$$

$\mathcal{H}_n$ satisfies:

1. If $f_{2-k,1} | \mathcal{H}$ is weakly holomorphic and $f_{k,1} | \mathcal{H}$ has integer coefficients, then

$$\left( f_{k,1} | \mathcal{H}_n \right) | \mathcal{H} \equiv f_{k,1} | \mathcal{H} \pmod{p^n}.$$  

2. If $\mathcal{H}_n$ and $\mathcal{H}_n'$ are two such operators, then

$$f_{k,1} | \mathcal{H}_n \equiv f_{k,1} | \mathcal{H}_n' \pmod{p^n}.$$  

3. If $\left( f_{k,1} | \mathcal{H}_n \right) | \mathcal{H} \equiv 0 + O(q) \pmod{p^m}$ for some $m \leq n$, then

$$\left( f_{k,1} | \mathcal{H}_n \right) | \mathcal{H} \equiv 0 \pmod{p^m}.$$
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Then is a Hecke operator $S_n$ such that

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$$f_{2-k,1}|\mathcal{H}_n \in M_{2-k}^!,$$  
$$f_{k,1}|\mathcal{H}_n \in \mathbb{Z}((q)), $$

and $f_{k,1}|\mathcal{H}_n \equiv q^{-1} + O(q) \pmod{p^n}$. Any such $\mathcal{H}_n$ satisfies:
Theorem (G-R)

Then is a Hecke operator $\mathcal{S}_n$ such that

$$f_{2-k,1}|\mathcal{S}_n \in M_{2-k}^!, \quad f_{k,1}|\mathcal{S}_n \in \mathbb{Z}((q)),$$

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1. If $f_{2-k,1}|H$ is weakly holomorphic and $f_{k,1}|H$ has integer coefficients, then
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$$

3. If $(f_{k,1}|\mathcal{H}_n)|H \equiv 0 + O(q) \pmod{p^m}$ for some $m \leq n$, then $(f_{k,1}|\mathcal{H}_n)|H \equiv 0 \pmod{p^m}$. 


Corollary

If \( f_{k,1} \mid H \) has integer coefficients, \( p \) is ordinary for all eigenforms in a basis of \( S_k \), and \( f_{k,1} \mid H \equiv 0 + O(q) \) (mod \( p^n \)), then
Corollary

If $f_{k,1}|H$ has integer coefficients, $p$ is ordinary for all eigenforms in a basis of $S_k$, and $f_{k,1}|H \equiv 0 + O(q) \pmod{p^n}$, then

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- The holomorphic part of $Z_D(f)$ has integral principal part.
Integrality of Coefficients

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- The holomorphic part of $\mathcal{Z}_D(f)$ has integral principal part.

- Use induction to extend the corollary to linear combinations.
Our Main Theorem

Theorem (G-R)

Let \( f(z) \in M^!_k, \ 0 > k \in 2\mathbb{Z} \) have integral principal part. Denote the \( n^{th} \) symmetric function in the singular moduli of discriminant \( d \) for \( \partial f \) by \( S_f(n; d) \). Let

\[
B(n, k) := \begin{cases} 
\frac{-nk}{4} & \text{if } nk \in 4\mathbb{Z} \\
\frac{1}{4}(-nk + 2k - 2) & \text{otherwise.}
\end{cases}
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Our Main Theorem

Theorem (G-R)

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$$B(n, k) := \begin{cases} \frac{-nk}{4} & \text{if } nk \in 4\mathbb{Z} \\ \frac{1}{4}(-nk + 2k - 2) & \text{otherwise.} \end{cases}$$

Then if $(p, d) = 1$, we have that $S_f(n; d)$ is $p$-integral. If $p|d$ is good for $(k, N)$, we have that

$$p^{B(n,k)} \cdot S_f(n; d) \text{ is } p\text{-integral.}$$