Bounding the Denominators of CM-values of Certain Weak Maass Form

Eric Larson and Larry Rolen

Harvard University and Emory University

The Partition Function

- A *partition* of a positive integer n is any nonincreasing sequence of positive integers which sum to n.

The Partition Function

- A *partition* of a positive integer n is any nonincreasing sequence of positive integers which sum to n.
- The partition function $p(n) =$ the number of partitions of n.

The Partition Function

- A *partition* of a positive integer n is any nonincreasing sequence of positive integers which sum to n.
- The partition function $p(n) =$ the number of partitions of n.
- Hardy and Ramanujan proved the asymptotitc

$$p(n) \sim \frac{1}{4n\sqrt{3}} \cdot e^{\pi\sqrt{2n/3}}.$$

Rademacher's Formula

- Rademacher later refined Hardy and Ramanujan's method to obtain an "exact formula" for $p(n)$:

Rademacher's Formula

- Rademacher later refined Hardy and Ramanujan's method to obtain an "exact formula" for $p(n)$:

Theorem (Rademacher)

$$p(n) = 2\pi(24n-1)^{-\frac{3}{4}} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} \cdot I_{\frac{3}{2}}\left(\frac{\pi\sqrt{24n-1}}{6k}\right).$$

A Finite Formula for $p(n)$

Theorem (Bruinier-Ono)

There exists a weak Maass form $P_p(z)$ such that

$$p(n) = \frac{1}{24n-1} \sum_{Q \in \mathcal{Q}_n} P_p(\alpha_Q).$$

For each n, this is a finite sum. Moreover, each $P_p(\alpha_Q)$ is algebraic.

A Natural Question of Bruinier and Ono

Remark

They also prove that

1. $6 \cdot (24n - 1) \cdot P(\alpha_Q)$ *is an algebraic integer.*

A Natural Question of Bruinier and Ono

Remark

They also prove that

1. $6 \cdot (24n - 1) \cdot P(\alpha_Q)$ *is an algebraic integer.*
2. *The numbers* $P(\alpha_Q)$*, as* Q *varies over* $\mathcal{Q}_n$*, form a multiset which is a union of Galois orbits for the discriminant* $-24n + 1$ *ring class field.*

A Natural Question of Bruinier and Ono

Remark

They also prove that

1. $6 \cdot (24n - 1) \cdot P(\alpha_Q)$ *is an algebraic integer.*
2. *The numbers* $P(\alpha_Q)$, *as* Q *varies over* $\mathcal{Q}_n$, *form a multiset which is a union of Galois orbits for the discriminant* $-24n + 1$ *ring class field.*

Conjecture (Bruinier-Ono)

We have that $(24n - 1) \cdot P(\alpha_Q)$ *is an algebraic integer.*

Our Results

Theorem (L-R 2011)

Suppose $F \in M^!_{-2}(\Gamma_0(N))$ is such that the Fourier expansions of

$$F \qquad \text{and} \qquad q\frac{dF}{dq} + F \cdot \frac{E_2E_4 - E_6}{6E_4}$$

at all cusps have coefficients that are algebraic integers. Let α_Q be a CM point of discriminant $-24n+1$, and let $P(z)$ be the weak Maass form

$$P(z) = -\left(\frac{1}{2\pi i} \cdot \frac{d}{dz} + \frac{1}{2\pi y}\right) F(z).$$

Then $(24n-1) \cdot P(\alpha_Q)$ is an algebraic integer.

Strategy of Proof

- Using the work of Bruinier and Ono, it suffices to show $P(\alpha_Q)$ is 6-integral.

Strategy of Proof

- Using the work of Bruinier and Ono, it suffices to show $P(\alpha_Q)$ is 6-integral.
- We write $P = A + B \cdot C$ where:

Strategy of Proof

- Using the work of Bruinier and Ono, it suffices to show $P(\alpha_Q)$ is 6-integral.
- We write $P = A + B \cdot C$ where:

$$A = -q\frac{dF}{dq} - \frac{1}{6}FE_2 + \frac{FE_6(7j - 6912)}{6E_4(j - 1728)},$$

$$B = \frac{FE_6 j}{E_4},$$

$$C = \frac{E_4}{6E_6 j}\left(E_2 - \frac{3}{\pi \operatorname{Im} z}\right) - \frac{7j - 6912}{6j(j - 1728)}.$$

Application to the Conjecture of Bruinier-Ono

- The level of F_p is 6, a square-free integer, so the Atkin-Lehner involutions act transitively on the cusps.

Application to the Conjecture of Bruinier-Ono

- The level of F_p is 6, a square-free integer, so the Atkin-Lehner involutions act transitively on the cusps.
- We have that F_p is an eigenform for the Atkin-Lehner involutions and has integral Fourier coefficients at infinity.

Application to the Conjecture of Bruinier-Ono

- The level of F_p is 6, a square-free integer, so the Atkin-Lehner involutions act transitively on the cusps.
- We have that F_p is an eigenform for the Atkin-Lehner involutions and has integral Fourier coefficients at infinity.
- Thus, F_p has integral Fourier coefficients at all cusps.

Application to the Conjecture of Bruinier-Ono

- The level of F_p is 6, a square-free integer, so the Atkin-Lehner involutions act transitively on the cusps.
- We have that F_p is an eigenform for the Atkin-Lehner involutions and has integral Fourier coefficients at infinity.
- Thus, F_p has integral Fourier coefficients at all cusps.
- The other condition in the theorem follows as Maass raising operators commute with Atkin-Lehner involutions.

Proof of 6-integrality of A, B

- We show that $j(\alpha_Q)$ is a unit at 2,3. This follows from the:

Proof of 6-integrality of A, B

- We show that $j(\alpha_Q)$ is a unit at 2,3. This follows from the:

Lemma (Deuring?)

Let $p \in \{2, 3\}$ and E be an elliptic curve defined over a number field K having CM by an order in a quadratic field F. If E has good ordinary reduction at all primes lying over p, then $j(E)$ is coprime to p.

Proof of 6-integrality of A, B (cont.)

- By the assumptions on F, $A \cdot j \cdot (j - 1728)$ and B are weakly holomorphic with integral Fourier expansions at all cusps.

Proof of 6-integrality of A, B (cont.)

- By the assumptions on F, $A \cdot j \cdot (j - 1728)$ and B are weakly holomorphic with integral Fourier expansions at all cusps.
- The 6-integrality of A, B now follows from the same argument as in Bruinier and Ono.

Classical Modular Polynomials

Definition

We say that two matrices B_1 *and* B_2 *are equivalent if* $B_1 = X \cdot B_2$ *for some* $X \in \mathrm{SL}_2(\mathbb{Z})$.

Classical Modular Polynomials

Definition

We say that two matrices B_1 and B_2 are equivalent if $B_1 = X \cdot B_2$ for some $X \in \mathrm{SL}_2(\mathbb{Z})$.

- There are finitely many equivalence classes of primitive integer matrices of determinant $-D$, which we call $M_1, M_2, \ldots, M_n$ with M_1 such that $\alpha_Q = M_1 \alpha_Q$.

Classical Modular Polynomials

Definition

We say that two matrices B_1 and B_2 are equivalent if $B_1 = X \cdot B_2$ for some $X \in \mathrm{SL}_2(\mathbb{Z})$.

- There are finitely many equivalence classes of primitive integer matrices of determinant $-D$, which we call $M_1, M_2, \ldots, M_n$ with M_1 such that $\alpha_Q = M_1 \alpha_Q$.

Definition

We write $\Phi_{-D}(X, Y)$ for the classical modular polynomial:

Classical Modular Polynomials

Definition

We say that two matrices B_1 and B_2 are equivalent if $B_1 = X \cdot B_2$ for some $X \in \mathrm{SL}_2(\mathbb{Z})$.

- There are finitely many equivalence classes of primitive integer matrices of determinant $-D$, which we call $M_1, M_2, \ldots, M_n$ with M_1 such that $\alpha_Q = M_1 \alpha_Q$.

Definition

We write $\Phi_{-D}(X, Y)$ for the classical modular polynomial:

$$\Phi_{-D}(j(z), Y) = \prod_{i=1}^{n} (Y - j(M_i z)).$$

Description of $C(\alpha_Q)$ using Modular Polynomials

- We expand $\Phi_{-D}(X, Y)$ in a power series about $X = Y = j(\alpha_Q)$ as

Description of $C(\alpha_Q)$ using Modular Polynomials

- We expand $\Phi_{-D}(X, Y)$ in a power series about $X = Y = j(\alpha_Q)$ as

$$\Phi(X, Y) = \sum_{\mu,\nu} \beta_{\mu,\nu}(X - j(\alpha_Q))^{\mu}(Y - j(\alpha_Q))^{\nu},$$

where $\beta_{\mu,\nu}$ is an algebraic integer. We write $\beta = \beta_{0,1} = \beta_{1,0}$.

Description of $C(\alpha_Q)$ using Modular Polynomials

- We expand $\Phi_{-D}(X, Y)$ in a power series about $X = Y = j(\alpha_Q)$ as

$$\Phi(X, Y) = \sum_{\mu,\nu} \beta_{\mu,\nu}(X - j(\alpha_Q))^{\mu}(Y - j(\alpha_Q))^{\nu},$$

where $\beta_{\mu,\nu}$ is an algebraic integer. We write $\beta = \beta_{0,1} = \beta_{1,0}$.

Theorem (Masser)

We have that $C(\alpha_Q) = \frac{\beta_{0,2} - \beta_{1,1} + \beta_{2,0}}{\beta}$.

Description of $C(\alpha_Q)$ using Modular Polynomials

- We expand $\Phi_{-D}(X, Y)$ in a power series about $X = Y = j(\alpha_Q)$ as

$$\Phi(X, Y) = \sum_{\mu,\nu} \beta_{\mu,\nu}(X - j(\alpha_Q))^{\mu}(Y - j(\alpha_Q))^{\nu},$$

where $\beta_{\mu,\nu}$ is an algebraic integer. We write $\beta = \beta_{0,1} = \beta_{1,0}$.

Theorem (Masser)

We have that $C(\alpha_Q) = \frac{\beta_{0,2} - \beta_{1,1} + \beta_{2,0}}{\beta}$.

- By definition, $\beta = \prod_{i=2}^{n}(j(\alpha_Q) - j(M_i\alpha_Q))$.

Proof of 6-integrality for $C(\alpha_Q)$

- It suffices to show that for any prime $\mathfrak{p}$ lying over 6, we have $j(\alpha_Q) \not\equiv j(M_i\alpha_Q) \bmod \mathfrak{p}$.

Proof of 6-integrality for $C(\alpha_Q)$

- It suffices to show that for any prime $\mathfrak{p}$ lying over 6, we have $j(\alpha_Q) \not\equiv j(M_i\alpha_Q) \bmod \mathfrak{p}$.
- This follows from the:

Proof of 6-integrality for $C(\alpha_Q)$

- It suffices to show that for any prime $\mathfrak{p}$ lying over 6, we have $j(\alpha_Q) \not\equiv j(M_i\alpha_Q) \bmod \mathfrak{p}$.
- This follows from the:

Lemma (Deuring?)

Suppose $\mathfrak{p}$ is a prime ideal of a number field K. Suppose E and E' are two elliptic curves over K with complex multiplication by the same order R in a quadratic field F. Suppose the index $[\mathcal{O}_F : R]$ is coprime to the residue characteristic of $\mathfrak{p}$. If both curves have good ordinary reduction at $\mathfrak{p}$ and the reduced curves are isomorphic, then E and E' are also isomorphic.

Conclusion

Theorem (L-R 2011)

Suppose $F \in M^!_{-2}(\Gamma_0(N))$ is such that the Fourier expansions of

$$F \quad \text{and} \quad q\frac{dF}{dq} + F \cdot \frac{E_2 E_4 - E_6}{6E_4}$$

at all cusps have coefficients that are algebraic integers. Let α_Q be a CM point of discriminant $-24n+1$, and let $P(z)$ be the weak Maass form

$$P(z) = -\left(\frac{1}{2\pi i} \cdot \frac{d}{dz} + \frac{1}{2\pi y}\right) F(z).$$

Then $(24n-1) \cdot P(\alpha_Q)$ is an algebraic integer.