Pullbacks of Siegel Eisenstein Series and Critical L-values

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- Dirichlet's Theorem on arithmetic progressions: non-vanishing of $L(1,\chi)$.

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- $\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty}((1 - q^n)$
- By Euler's identity, $\sum_{n=1}^{\infty} p(n)q^{24n-1} = \frac{1}{\eta(24z)}$

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- Analytic continuation to $\mathbb{C}$, functional equations, etc.

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- $n = 1$ corresponds to modular forms

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- Fourier coefficients involve L-functions, generalized class numbers, etc.

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- $$\sum_{f \in \mathcal{B}_k} \frac{L(k-1, f, \mathrm{St}) A_k(f)}{\langle f, f \rangle} = \alpha_k$$
- For $k \in \{12, 16, 18, 20, 22\}$, $\dim(S_k) = 1$

Table: Values of α_k

k	α_k
$k = 12$	$(2^{31} \cdot \pi^{33})/(3^6 \cdot 5^3 \cdot 7^3 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 691)$
$k = 14$	0
$k = 16$	$(2^{40} \cdot \pi^{45})/(3^{13} \cdot 5^6 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 3617)$
$k = 18$	$(2^{37} \cdot \pi^{51})/(3^{12} \cdot 5^5 \cdot 7^5 \cdot 11^3 \cdot 13^2 \cdot 17^2 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 43687)$
$k = 20$	$(2^{39} \cdot \pi^{57})/(3^{17} \cdot 5^7 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot 29 \cdot 31 \cdot 37 \cdot 283 \cdot 617)$
$k = 22$	$(2^{42} \cdot 4409 \cdot \pi^{63})/(3^{21} \cdot 5^8 \cdot 7^5 \cdot 11^3 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 131 \cdot 593)$

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- Choose convenient basis, use inner product formula due to Shimura

Sketch of Proof (cont.)

- $E_k^*(z,w) = E_{1,k}(z)E_{2,k}(w) + c_k \sum_{f\in\mathcal{B}_k} \frac{L(k-1,f,\mathrm{St})}{\langle f,f\rangle} f(z)(E_{1,k}^2(f))^c(w)$

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- $-\det(2T)$ is a fundamental discriminant for the other 3, formula of Mizumoto applies.

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- Further problems: Higher degree, arbitrary level
- Hilbert modular forms: Much of the theory carries through in general (totally real) number fields