Mock theta functions and quantum modular forms

Larry Rolen

University of Cologne

This research was supported by the University of Cologne and the DFG.

June 2, 2015

Ramanujan's "Deathbed" letter

- In 1920, Ramanujan gave a 17 "Eulerian series", such as

$$f(q) := \sum_{n \geq 0} \frac{q^{n^2}}{(-q)_n^2},$$

where $(a;q)_n := (a)_n = \prod_{j=0}^{n-1} \left(1 - aq^j\right)$.

Ramanujan's "Deathbed" letter

- In 1920, Ramanujan gave a 17 "Eulerian series", such as

$$f(q) := \sum_{n\geq 0} \frac{q^{n^2}}{(-q)_n^2},$$

where $(a;q)_n := (a)_n = \prod_{j=0}^{n-1}\left(1-aq^j\right)$.

Question

What is the relation of functions like $f(q)$ to modular forms?

Ramanujan's "Deathbed" letter

- In 1920, Ramanujan gave a 17 "Eulerian series", such as

$$f(q) := \sum_{n\geq 0} \frac{q^{n^2}}{(-q)_n^2},$$

where $(a;q)_n := (a)_n = \prod_{j=0}^{n-1} \left(1 - aq^j\right)$.

Question

What is the relation of functions like $f(q)$ to modular forms?

- Example: The Rogers-Ramanujan identities

$$G(q) := \sum_{n\geq 0} \frac{q^{n^2}}{(q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty},$$

$$H(q) := \sum_{n\geq 0} \frac{q^{n^2+n}}{(q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}.$$

Which Eulerian series are modular?

- The functions G and H are essentially modular functions.

Which Eulerian series are modular?

- The functions G and H are essentially modular functions.

Question

How does one "detect" modularity of an Eulerian series?

Which Eulerian series are modular?

- The functions G and H are essentially modular functions.

Question

How does one "detect" modularity of an Eulerian series?

- Modular forms have very strong properties in their asymptotic expansions!

Which Eulerian series are modular?

- The functions G and H are essentially modular functions.

Question

How does one "detect" modularity of an Eulerian series?

- Modular forms have very strong properties in their asymptotic expansions!
- For example, a modular form must have an asymptotic expansion as $t \to 0^+$ of the shape

$$e^{\frac{a}{t}} F(e^{-t}) \sim bt^{-k} + O(t^N) \text{ for all } N \geq 0,$$

but most Eulerian series have "unclosed" expansions.

The mock theta functions

- Ramanujan noticed that all of his mock theta functions "look like" modular forms near roots of unity, and he defined mock theta functions by this property.

The mock theta functions

- Ramanujan noticed that all of his mock theta functions "look like" modular forms near roots of unity, and he defined mock theta functions by this property.

Example (Ramanujan-Watson)

Let $b(q) := (q)_\infty/(-q)_\infty^2$. Then if ζ is a primitive $2k$-th order root of unity,

$$\lim_{q\to\zeta}\left(f(q)-(-1)^k b(q)\right) = O(1).$$

The mock theta functions

- Ramanujan noticed that all of his mock theta functions "look like" modular forms near roots of unity, and he defined mock theta functions by this property.

Example (Ramanujan-Watson)

Let $b(q) := (q)_\infty/(-q)_\infty^2$. Then if ζ is a primitive $2k$-th order root of unity,

$$\lim_{q\to\zeta}\Big(f(q)-(-1)^k b(q)\Big) = O(1).$$

- Hence, at even order roots of unity, the singularities of $f(q)$ are "cut out" by $\pm b(q)$. Mock theta functions are defined to be functions which have their singularities cut out by modular forms, but not in a trivial way.

From Ramanujan to the future, and back

- Zwegers, Bruinier-Funke: Ramanujan's mock theta functions are pieces of harmonic Maass forms.

From Ramanujan to the future, and back

- Zwegers, Bruinier-Funke: Ramanujan's mock theta functions are pieces of harmonic Maass forms.
- This was a huge revelation which led to an explosion of applications to many areas of math.

From Ramanujan to the future, and back

- Zwegers, Bruinier-Funke: Ramanujan's mock theta functions are pieces of harmonic Maass forms.
- This was a huge revelation which led to an explosion of applications to many areas of math.
- However, as noted by Berndt, no one had proven that Ramanujan's function satisfied his own definition.

From Ramanujan to the future, and back

- Zwegers, Bruinier-Funke: Ramanujan's mock theta functions are pieces of harmonic Maass forms.
- This was a huge revelation which led to an explosion of applications to many areas of math.
- However, as noted by Berndt, no one had proven that Ramanujan's function satisfied his own definition.

Theorem (Griffin-Ono-R.)

Ramanujan's original formulation of the mock theta functions was correct.

Looking further into Ramanujan's definition

Question

Can we understand Ramanujan's question more explicitly? Namely, can we provide an algorithm to systematically compute the modular forms to cut out the singularities, along with the "leftover constants"?

Looking further into Ramanujan's definition

Question

Can we understand Ramanujan's question more explicitly? Namely, can we provide an algorithm to systematically compute the modular forms to cut out the singularities, along with the "leftover constants"?

Example

Folsom, Ono, and Rhoades proved that if ζ is a primitive $2k$-th order root of unity,

$$\lim_{q \to \zeta} \left(f(q) - (-1)^k b(q) \right) = -4 \sum_{n=0}^{k-1} (-\zeta;\zeta)_n^2 \zeta^{n+1}.$$

Looking further into Ramanujan's definition

Question

Can we understand Ramanujan's question more explicitly? Namely, can we provide an algorithm to systematically compute the modular forms to cut out the singularities, along with the "leftover constants"?

Example

Folsom, Ono, and Rhoades proved that if ζ is a primitive $2k$-th order root of unity,

$$\lim_{q\to\zeta}\Big(f(q)-(-1)^k b(q)\Big) = -4\sum_{n=0}^{k-1}(-\zeta;\zeta)_n^2\zeta^{n+1}.$$

Moreover, they fit this into an infinite family of relations beautifully connecting the rank, crank, and unimodal generating functions.

"Universal" families

Idea (Rhoades)

Study Ramanujan's definition for:

$$g_2(\zeta;q) := \sum_{n\geq 0} \frac{(-q)_n q^{\frac{n(n+1)}{2}}}{(\zeta)_{n+1}(\zeta^{-1}q;q)_{n+1}}$$

“Universal” families

Idea (Rhoades)

Study Ramanujan’s definition for:

$$g_2(\zeta;q) := \sum_{n\geq 0} \frac{(-q)_n q^{\frac{n(n+1)}{2}}}{(\zeta)_{n+1}(\zeta^{-1}q;q)_{n+1}}$$

Goal

Find $f_{a,b,A,B,h,k}(q) \in M^!_{\frac{1}{2}}$ *and finite sums* $U_{a,b,A,B,h,k}$ *such that*

$$\lim_{q\to e^{2\pi i \frac{h}{k}}} \left(g_2\left(\zeta_b^a q^A; q^B\right) - f_{a,b,A,B,h,k}(q)\right) = U_{a,b,A,B,h,k}.$$

"Universal" families

Idea (Rhoades)

Study Ramanujan's definition for:

$$g_2(\zeta; q) := \sum_{n \geq 0} \frac{(-q)_n q^{\frac{n(n+1)}{2}}}{(\zeta)_{n+1}(\zeta^{-1}q; q)_{n+1}}$$

Goal

Find $f_{a,b,A,B,h,k}(q) \in M^{!}_{\frac{1}{2}}$ *and finite sums* $U_{a,b,A,B,h,k}$ *such that*

$$\lim_{q \to e^{2\pi i \frac{h}{k}}} \left(g_2\left(\zeta_b^a q^A; q^B\right) - f_{a,b,A,B,h,k}(q) \right) = U_{a,b,A,B,h,k}.$$

Theorem (Bringmann-R.)

There is a canonical, finite procedure to solve this problem. At most three functions $f_{a,b,A,B,h,k}$ *are needed for fixed* a, b, A, B.

Quantum modular forms

"Definition"

A quantum modular form is a function $f : \mathbb{P}^1(\mathbb{Q}) \to \mathbb{C}$ *such that for all* $\gamma \in \Gamma$, $f|_k(1-\gamma)$ *is "nice".*

Quantum modular forms

"Definition"

A quantum modular form is a function $f : \mathbb{P}^1(\mathbb{Q}) \to \mathbb{C}$ *such that for all* $\gamma \in \Gamma$, $f|_k(1-\gamma)$ *is "nice".*

Example

Kontsevich defined:

$$F(q) := \sum_{n \geq 0} (q)_n.$$

This converges at roots of unity, and its values equal the radial limits of a "half-derivative" of the Dedekind eta function. Zagier used this to show that F is a QMF of weight $\frac{3}{2}$.

Quantum modular forms

"Definition"

A quantum modular form is a function $f : \mathbb{P}^1(\mathbb{Q}) \to \mathbb{C}$ *such that for all* $\gamma \in \Gamma$, $f|_k(1-\gamma)$ *is "nice".*

Example

Kontsevich defined:

$$F(q) := \sum_{n \geq 0} (q)_n.$$

This converges at roots of unity, and its values equal the radial limits of a "half-derivative" of the Dedekind eta function. Zagier used this to show that F is a QMF of weight $\frac{3}{2}$.

Theorem (Choi-Lim-Rhoades)

If f *is a mock theta function, then the "leftover constants" in Ramanujan's definition give a quantum modular form.*

Further examples of quantum modular forms

Theorem (Bringmann-R.)

The "Eichler integral" (the formal $k-1$st antiderivative) of any half-integral weight cusp form is a quantum modular form.

Further examples of quantum modular forms

Theorem (Bringmann-R.)

The "Eichler integral" (the formal $k-1$st antiderivative) of any half-integral weight cusp form is a quantum modular form.

- These quantum modular forms show up all over the place! Call them Eichler quantum modular forms.

Further examples of quantum modular forms

Theorem (Bringmann-R.)

The "Eichler integral" (the formal $k-1$st antiderivative) of any half-integral weight cusp form is a quantum modular form.

- These quantum modular forms show up all over the place! Call them Eichler quantum modular forms.

Theorem (Bringmann-Creutzig-R., Bringmann-R.-Zwegers)

The Fourier coefficients in z of a negative index Jacobi form have "theta-type" expansions in terms of quasimodular forms and Eichler quantum modular forms.

Further examples of quantum modular forms

Theorem (Bringmann-R.)

The "Eichler integral" (the formal $k-1$st antiderivative) of any half-integral weight cusp form is a quantum modular form.

- These quantum modular forms show up all over the place! Call them Eichler quantum modular forms.

Theorem (Bringmann-Creutzig-R., Bringmann-R.-Zwegers)

The Fourier coefficients in z of a negative index Jacobi form have "theta-type" expansions in terms of quasimodular forms and Eichler quantum modular forms.

Goal (Bringmann-R.)

Understand the general framework of quantum modular forms, for example by starting with a well-defined subspace such as the Eichler quantum modular forms.

Arithmetic properties of quantum modular forms

The function F has a Taylor expansion at $q = 1$ given by

$$\sum_{n\geq 0}(1-q;1-q)_n =: \sum_{n\geq 0}\xi(n)q^n = 1+q+2q^2+5q^3+\dots.$$

Arithmetic properties of quantum modular forms

The function F has a Taylor expansion at $q = 1$ given by

$$\sum_{n\geq 0}(1-q;1-q)_n =: \sum_{n\geq 0}\xi(n)q^n = 1 + q + 2q^2 + 5q^3 + \dots.$$

Theorem (Andrews-Sellers,Straub)

There are infinitely many primes p for which there is a $B \in \mathbb{N}$ such that for all A,

$$\xi\left(p^A n - B\right) \equiv 0 \pmod{p^A}.$$

The function F has a Taylor expansion at $q = 1$ given by

$$\sum_{n\geq 0}(1-q;1-q)_n =: \sum_{n\geq 0}\xi(n)q^n = 1 + q + 2q^2 + 5q^3 + \dots .$$

Theorem (Andrews-Sellers,Straub)

There are infinitely many primes p for which there is a $B \in \mathbb{N}$ such that for all A,

$$\xi\left(p^A n - B\right) \equiv 0 \pmod{p^A}.$$

Theorem (Guerzhoy-Kent-R.)

For any weight $1/2$ theta function, there are analogous sequences defined by Taylor expansions of the associated Eichler quantum modular form. Moreover, these (almost always) satisfy congruences like those for $\xi(n)$ for a positive proportion of primes.

Motivating example from knot theory

- Hikami considered

$$F_m^{(\alpha)}(q) := \sum_{k_1,k_2,\ldots,k_m=0}^{\infty} (q;q)_{k_m} q^{k_1^2+\ldots+k_{m-1}^2+k_\alpha+\ldots+k_{m-1}}$$

$$\times \left(\prod_{\substack{i=1\\ i\neq\alpha}}^{m-1} \begin{bmatrix} k_i+1 \\ k_i \end{bmatrix}_q \right) \cdot \begin{bmatrix} k_{\alpha+1}+1 \\ k_\alpha \end{bmatrix}_q ,$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \begin{cases} \frac{(q)_n}{(q)_k (q)_{n-k}} & 0 \leq k \leq n \\ 0 & \text{otherwise.} \end{cases}$$

Motivating example from knot theory

- Hikami considered

$$F_m^{(\alpha)}(q) := \sum_{k_1,k_2,\dots,k_m=0}^{\infty} (q;q)_{k_m} q^{k_1^2+\dots+k_{m-1}^2+k_\alpha+\dots+k_{m-1}}$$

$$\times \left(\prod_{\substack{i=1\\ i\neq\alpha}}^{m-1} \begin{bmatrix} k_i+1 \\ k_i \end{bmatrix}_q \right) \cdot \begin{bmatrix} k_{\alpha+1}+1 \\ k_\alpha \end{bmatrix}_q,$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \begin{cases} \frac{(q)_n}{(q)_k(q)_{n-k}} & 0 \leq k \leq n \\ 0 & \text{otherwise.} \end{cases}$$

- These are limits of the "half-derivative" of Andrews-Gordon functions, and related to Kashaev's invariant for torus knots.

Explicit form of congruences for Hikami's examples

Define numbers $\xi_m^{(a)}$ as the Taylor coefficients of $F_m^{(a)}$ at $q = 1$.

Explicit form of congruences for Hikami's examples

Define numbers $\xi_m^{(a)}$ as the Taylor coefficients of $F_m^{(a)}$ at $q = 1$.

Theorem (Guerzhoy-Kent-R.)

Choose $\alpha, m \in \mathbb{N}$ *with* $\alpha < m$ *such that* $(2m - 2\alpha - 1)^2 - 8(2m + 1)$ *is not a square. Then*

$$\xi_m^{(\alpha)}\left(p^A n - 1\right) \equiv 0 \pmod{p^A}$$

for all $n, A \in \mathbb{N}$ *for at least* 50% *of primes* p.