Zeta-polynomials for modular form periods

Larry Rolen

February 25, 2016

Riemann's zeta-function

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$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

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2. *We have the* **functional equation**

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{(1-s)}{2}} \cdot \Gamma\left(\frac{1-s}{2}\right) \cdot \zeta(1-s).$$

\$1 million prize problem

Conjecture (Riemann)

Apart from the negative evens, the zeros of $\zeta(s)$ *satisfy* $\mathrm{Re}(s) = \frac{1}{2}$.

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1. *The "line of symmetry" for* $s \longleftrightarrow 1 - s$ *is* $\mathrm{Re}(s) = \frac{1}{2}$.
2. *The first "gazillion" zeros satisfy RH (Odlyzko).*
   Over 40% of the zeros satisfy RH (Selberg, Levinson, Conrey).

The values $\zeta(-n)$

Theorem (Euler)

As a power series in t*, we have*

$$\frac{t}{1-e^{-t}} = 1 + \frac{1}{2}t - t\sum_{n=1}^{\infty} \zeta(-n) \cdot \frac{t^n}{n!}.$$

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Remark

This series is essentially the generating function for K-groups of $\mathbb{Q}$.

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Theorem (Main Theorem)

Manin's Conjecture is true.

Fundamental Theorem for modular L-functions

Theorem (Hecke, Atkin-Lehner, Shimura, Manin, and others)

If $f \in S_k(\Gamma_0(N))$ is a newform, then the following are true:

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3. *There are numbers* $\omega_f^{\pm}$ *such that for* $1 \leq j \leq k-1$

$$L(f,j) \in \overline{\mathbb{Q}} \cdot (2\pi i)^j \cdot \omega_f^{\pm}.$$

Critical Values and Weighted Moments

Definition (Manin, Shimura)

If $f \in S_k(\Gamma_0(N))$ is a newform, then its **critical L-values** are

$$\{L(f,1), \quad L(f,2), \quad L(f,3), \ldots, \quad L(f,k-1)\}.$$

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Definition (Ono-R-Sprung)

If $m \geq 1$, then we define the **weighted moments**

$$M_f(m) := \frac{1}{(k-2)!}\sum_{j=0}^{k-2}\binom{k-2}{j}\Lambda(f,j+1)\cdot j^m.$$

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where the (signed) Stirling numbers of the first kind are given by

$$(x)_n = x(x-1)(x-2)\cdots(x-n+1) =: \sum_{m=0}^{n} s(n,m) x^m.$$

The $s(n,k)$ form Pascal-type triangles

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 & & & & & & 1 & & & & & & \\
 & & & & & 0 & & 1 & & & & & \\
 & & & & 0 & & -1 & & 1 & & & & \\
 & & & 0 & & 2 & & -3 & & 1 & & & \\
 & & 0 & & -6 & & 11 & & -6 & & 1 & & \\
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Remark

$Z_f(s)$ is a cobbling of layers of these weighted by moments $M_f(m)$.

Functional Equations and the Riemann Hypothesis

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If $f \in S_k(\Gamma_0(N))$ *is an even weight* $k \geq 4$ *newform, then we have:*

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- *The $Z(-n)$ encode arithmetic-geometric information.*

Example of $\Delta \in S_{12}$

$$Z_\Delta(s) \approx (5.11 \times 10^{-7}) s^{10} + \cdots - 0.0199 s + 0.00596.$$

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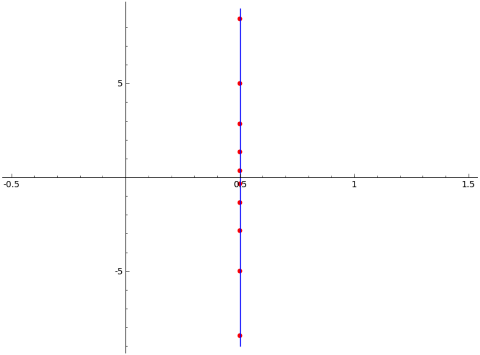

Figure: The roots of $Z_\Delta(s)$

A Nice Generating Function

Theorem 2 (Ono-R-Sprung)

Define the **normalized period polynomial** *for f by*

$$R_f(z) := \sum_{j=0}^{k-2} \binom{k-2}{j} \cdot \Lambda(f, k-1-j) \cdot z^j.$$

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Remark (Euler)

$$\frac{t}{1-e^{-t}} = 1 + \frac{1}{2}t - t\sum_{n=1}^{\infty} \zeta(-n) \cdot \frac{t^n}{n!}.$$

Arithmetic Geometric Information

Conjecture (Bloch–Kato). *Let* $0 \leq j \leq k-2$, *and assume* $L(f, j+1) \neq 0$. *Then we have*

$$\frac{L(f, j+1)}{(2\pi i)^{j+1}\Omega^{(-1)^{j+1}}} = u_{j+1} \times \frac{\mathrm{Tam}(j+1)\#\text{Ш}(j+1)}{\#H^0_{\mathbb{Q}}(j+1)\#H^0_{\mathbb{Q}}(k-1-j)} =: C(j+1)$$

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Corollary (Ono-R-Sprung)

Assuming the Bloch-Kato Conjecture, we have that

$$M_f(m) = \sum_{0 \leq j \leq k-2} \widetilde{C(j+1)} j^m.$$

Combinatorial Polynomials $H_k^{\pm}(s)$

Definition (Binomial Coefficient)

If $x, y \in \mathbb{C}$, then the complex **binomial coefficient** $\binom{x}{y}$ is

$$\binom{x}{y} := \frac{\Gamma(x+1)}{\Gamma(y+1)\Gamma(x-y+1)}.$$

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Definition (Special Polynomials)

If $k \geq 4$ is even, then

$$H_k^+(s) := \binom{s+k-2}{k-2} + \binom{s}{k-2},$$

$$H_k^-(s) := \sum_{j=0}^{k-3} \binom{s-j+k-3}{k-3}.$$

The $H_k^{\pm}(-s)$ Approximate $Z_f(s)$

Theorem 3 (Ono-R-Sprung)

Suppose that $k \geq 4$ *and* $\epsilon \in \{\pm 1\}$. *Then we have that*

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where $f \in S_k(\Gamma_0(N))$ *are chosen with* $\epsilon(f) = \epsilon$.

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Remark

This offers an unexpected connection to polytopes.

Ehrhart Polynomials

Definition

Given a d-dimensional integral lattice polytope in $\mathbb{R}^n$, the **Ehrhart polynomial** $\mathcal{L}_p(x)$ is determined by

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Example

The polynomials $H_k^-(s)$ are the Ehrhart polynomials of the simplex

$$\operatorname{conv}\left\{e_1, e_2, \ldots, e_{k-3}, -\sum_{j=1}^{k-3} e_j\right\}.$$

Limits of $f \in S_6(\Gamma_0(N))$ with $\epsilon(f) = -1$

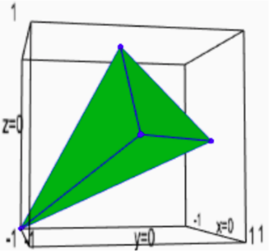

Figure: The tetrahedron whose Ehrhart polynomial is $H_6^-(s)$.

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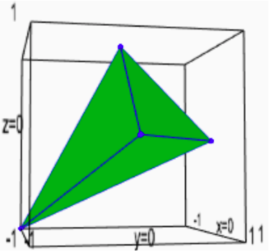

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$$\lim_{N\to+\infty} Z_f(s)$$

$$= H_6^-(-s) = -\frac{2}{3}\left(s-\frac{1}{2}\right)\left(s-\frac{1}{2}+\frac{\sqrt{-11}}{2}\right)\left(s-\frac{1}{2}-\frac{\sqrt{-11}}{2}\right).$$

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Then we have that

$$\frac{R_f(z)}{(1-z)^{k-1}} = \sum_{n=0}^{\infty} Z_f(-n) z^n.$$

Theorem (Rodriguez-Villegas (2002))

Suppose that $U(z) \in \mathbb{R}[z]$ is a degree e polynomial with $U(1) \neq 0$. Then there is a polynomial $H(z)$ for which

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If all roots of $U(z)$ *are on* $|z| = 1$*, then we have:*

1. *All roots of* $Z(s) := H(-s)$ *lie on* $\mathrm{Re}(z) = 1/2$*.*
2. *We have that*
$$Z(1-s) = \pm Z(s).$$

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Sketch of the proof of Theorems 1 and 2.

- For even weight $k \geq 4$ newforms f we **must prove** that

$$R_f(\rho) = 0 \quad \Longrightarrow \quad |z| = 1.$$

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- Make the definition of $Z_f(s) := H(-s)$ explicit (i.e. Stirling numbers and weight moments). □

□

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Problems (Open)

1. *Determine the* $r_f(X)$.
2. *Study the "distribution" of the zeros of* $r_f(X)$.

Example. $f \in S_4(\Gamma_0(8))$

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Let $f(\tau) = q - 4q^3 - 2q^5 + \cdots \in S_4(\Gamma_0(8))$ be the unique newform.

1. We find numerically that

$$L(f,1) \approx 0.354500683730965,$$
$$L(f,2) \approx 0.690031163123398,$$
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"Riemann Hypothesis" for Period Polynomials

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Remark

The circle $|z| = \frac{1}{\sqrt{N}}$ *is the "symmetry" for a functional equation.*

Previous Work

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- El-Guindy and Raji proved the $N = 1$ case.

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If $f \in S_k(\Gamma_0(N))$ is an even weight $k \geq 4$ newform, then all of the zeros of $R_f(z)$ satisfy $|z| = 1$.

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In particular, Theorems 1 and 2 are true.

Equidistribution

Theorem 5 (Jin-Ma-Ono-Soundararajan)

For fixed $\Gamma_0(N)$, as $k \to +\infty$, the zeros of $r_f(X) = 0$ become equidistributed on the circle with radius $\frac{1}{\sqrt{N}}$.

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Question

Can one do better than equidistribution?

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Remarks

- *For fixed k, the roots of $r_f(X)$ converge as $N \to +\infty$.*
- *This proves Theorem 3 that for fixed $\epsilon(f) \in \{\pm\}$ we have*

$$\lim_{N \to +\infty} Z_f(s) = H_k^{\pm}(-s).$$

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- This means that $\Lambda(f,3) \geq \Lambda(f,2)$. $\square$

Analytic Definition of $r_f(X)$

Lemma

If $f \in S_k(\Gamma_0(N))$ is a newform, then

$$r_f(X) = -\frac{(2\pi i)^{k-1}}{(k-2)!} \cdot \int_0^{i\infty} f(\tau)(\tau - X)^{k-2} d\tau.$$

$\mathrm{PSL}_2(\mathbb{R})^+$ action

Definition

If $\phi(z) \in \mathbb{C}[z]$ with $\deg(\phi) \leq k-2$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbb{R})^+$, then

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Remark

This defines a "modular action" on

$$V_{k-2} := \{\phi \in \mathbb{C}[z] \ : \ \deg(\phi) \leq k-2\}.$$

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$$r_f|(1 \pm W_N) = 0.$$

General Strategy

1. Let $m := \frac{k-2}{2}$, and define

$$P_f(X) := \frac{1}{2}\binom{2m}{m}\Lambda\left(f, \frac{k}{2}\right) + \sum_{j=1}^{m}\binom{2m}{m+j}\Lambda\left(f, \frac{k}{2}+j\right)X^j.$$

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$$T_f(X) := P_f(X) + \epsilon(f)P_f(1/X).$$

3. Letting $X \to z = e^{i\theta}$ on $|z| = 1$, then $T_f(z)$ is a "trigonometric" polynomial in sin or cos depending $\epsilon(f)$.

Classical Theorem of Pólya and Szegö

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Theorem (Szegö, 1936)

Suppose that $u(\theta)$ and $v(\theta)$ are

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If $0 \leq a_0 \leq a_1 \leq a_2 \cdots \leq a_{n-1} < a_n$, then both u and v have exactly n zeros in $[0, \pi)$, and these zeros are simple.

Useful inequalities

Lemma 1

The completed L-function $\Lambda(f, s)$ satisfies the following:

1) It is **monotone increasing** *in the range* $s \geq \frac{k}{2} + \frac{1}{2}$.

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3) If $\epsilon(f) = -1$, then $\Lambda\left(f, \frac{k}{2}\right) = 0$ and

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- Check remaining cases using SAGE.

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2. *The* $Z_f(-n)$ *encode the "Bloch-Kato complex."*
3. *For fixed* k *and* $\epsilon(f) = \epsilon$, *we have*

$$\lim_{N \to +\infty} Z_f(s) = H_k^{\epsilon}(-s).$$

This makes use of the following new result.

Theorem 4 (Jin-Ma-Ono-Soundararajan)

The Riemann Hypothesis for period polynomials is true.