LECTURE 2: CROSS PRODUCTS, MULTILINEARITY, AND AREAS OF PARALLELOGRAMS

MA1111: LINEAR ALGEBRA I, MICHAELMAS 2016

1. Finishing up dot products

Last time we stated the following theorem, for which I owe you the proof.

Theorem. *For any vectors v, w, w_1, w_2, and any $c \in \mathbb{R}$, we have the following.*

(1) $v \cdot w = w \cdot v$.
(2) $v \cdot (w_1 + w_2) = v \cdot w_1 + v \cdot w_2$.
(3) $v \cdot (cw) = c(v \cdot w)$.
(4) $v \cdot v = |v|^2$, *where $|v|$ denotes the length of v (also called the magnitude or norm).*
(5) $v \cdot w = |v||w| \cos \vartheta$, *where ϑ is the angle between v and w.*

Proof. (1)-(3) are all basic instances of the first basic strategy of proof writing. That is, give things names and start writing them down! We do this for (1) as an example leaving (2)-(3) as (extremely similar) exercises.

Proof of (1): We write $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$. Then by definition and a bit of rewriting we have

$$v \cdot w = v_1 w_1 + \ldots v_n w_n = w_1 v_1 + \ldots w_n v_n = w \cdot v.$$

N.B.: Part of the strategy of learning to find and remember proofs is to think in terms of the skeleton of key ideas. Most proofs, even if they look long, boil down to just a few main points. In this case, the reason that the dot product is commutative is simple: the real numbers are!

Proof of (4): This is simply a reformulation of the Pythagorean theorem. Namely,

$$v \cdot v = v_1^2 + v_2^2 + \ldots + v_n^2,$$

which is the square of the distance from the origin to the point $(v_1, v_2, \ldots, v_n)$ (think about it in 2 or 3 dimensions, for example).

Proof of (5): The name of the game in this case is the Law of Cosines. I hope most of you have seen it, but its not too difficult to state. Consider a triangle labelled as follows

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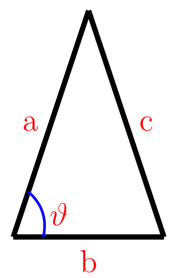
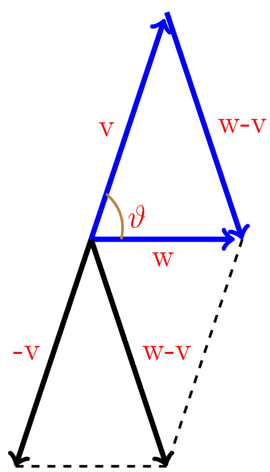


Then the Law of Cosines states that $c^2 = a^2 + b^2 - 2ab\cos\vartheta$, or equivalently

$$ab\cos\vartheta = \frac{1}{2}\left(a^2 + b^2 - c^2\right).$$

We just need to find an appropriate triangle to use. Consider the following diagram



Thus we can take a triangle as above with $a = |v|$, $b = |w|$, and $c = |w - v|$, and with an angle of the same name ϑ. Thus, we have

$$|v||w|\cos\vartheta = \frac{1}{2}\left(|v|^2 + |w|^2 - |w - v|^2\right).$$

By part (4), we can write this as

$$\begin{aligned}
&\frac{1}{2}\left(v_1^2 + \dots v_n^2 + w_1^2 + \dots + w_n^2 - (w_1 - v_1)^2 - \dots - (w_n - v_n)^2\right) \\
&= \frac{1}{2}\left([v_1^2 + w_1^2 - (v_1^2 + w_1^2 - 2v_1w_1)] + \dots + [v_n^2 + w_n^2 - (v_n^2 + w_n^2 - 2v_nw_n)]\right) \\
&= v_1w_1 + \dots v_nw_n,
\end{aligned}$$

which is $v \cdot w$ by definition. □

Part (5) is particularly useful as it allows one to find the angle between two vectors, or lines, in a simple algebraic way. Here is a particular instance which we will often use.

Corollary. *Two vectors v and w are orthogonal (perpendicular), written $v \perp w$, if and only if $v \cdot w = 0$.*

N.B.: "Corollary" is just another way of saying "theorem", but one which is a straightforward deduction of another "bigger" result.

2. Cross product

There is one more operation, which is a companion of the dot product, which is often useful. This one has a much less intuitive definition, and it is far from obvious as to why we should care about this operation. However, we take it as a given for now, noting that in particular it is very unique in that it has a number of nice properties (see the Theorem below) which are very hard to find functions satisfying. In fact, they are so hard to find, that this distinguished thing **only works in three dimensions.**

Definition. The cross product (or vector product) of two three-dimensional vectors v and w is given as the following vector, specified by its coordinates:

$$v \times w = (v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1).$$

Of course, there are a number of symmetries which make it easier to remember this definition. Note that each component of $v \times w$ depends on the *other* two components of v and w (for example, the second component depends on the first and third components of v, w). We will learn a slicker definition for this later in terms of other objects we don't know about yet, but for now this cumbersome definition will have to do.

Example. *We will often use the standard basis vectors*

$$i = (1, 0, 0), \qquad j = (0, 1, 0), \qquad k = (0, 0, 1).$$

The values of $\times$ on these vectors are as follows:

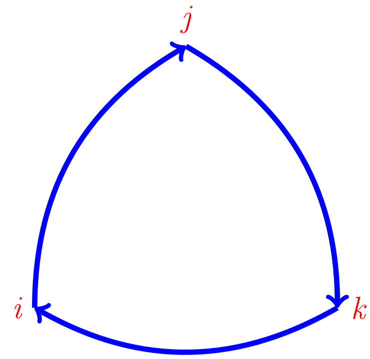
$$i \times i = j \times j = k \times k = 0,$$

$$i \times j = -j \times i = k,$$

$$i \times k = -k \times i = -j,$$

$$j \times k = -k \times j = i.$$

This is often remembered using the following mnemonic device, where the cross product of any two distinct standard basis vectors is the third one left out, with sign $+$ if it goes with the arrow and sign $-$ if it goes against the arrow:



We will see a little bit later in this lecture that the values on the standard basis vectors in fact easily determine the values everywhere.

The basic properties of $\times$ which make it of interest are as follows.

Theorem. *For any vectors u, v, w, w_1, w_2, and any $c \in \mathbb{R}$, the following hold.*

(1) $v \times w = -w \times v$.
(2) $v \times (w_1 + w_2) = v \times w_1 + v \times w_2$.
(3) $v \times (cw) = c(v \times w)$.
(4) $v \times v = 0$.
(5) $u \cdot (v \times w) = -v \cdot (u \times w)$.
(6) $u \times (v \times w) = (u \cdot w)v - (u \cdot v)w$.

Remark. *Although we have used the names u, v, w for consistency, there is a helpful mnemonic for property (6), the vector triple product, which almost looks like the non-sensical phrase "back cab":*

$$a \times (b \times c) = b(a \cdot c) - c(a \cdot b).$$

The cross product is neither commutative (see (1) above), nor associative (this is why the parentheses are required in (6); order matters). However, it is somewhat nearly commutative (it is what we call anticommutative because of the $-$ sign), and there is a replacement for associativity, called the Jacobi identity, which it does satisfy. At any rate, the lack of these properties holding means that many things you do in ordinary arithmetic as a matter of routine are no longer valid.

Proof. Parts (1)-(4) are all straightforward applications of the definition (see the similar proof in our big theorem on properties of the dot product).

Part (5) follows by first computing:

$$u \cdot (v \times w) = u_1(v_2w_3 - v_3w_2) + u_2(v_3w_1 - v_1w_3) + u_3(v_1w_2 - v_2w_1),$$

$$v \cdot (u \times w) = v_1(u_2w_3 - u_3w_2) + v_2(u_3w_1 - u_1w_3) + v_3(u_1w_2 - u_2w_1).$$

Adding these two equations, we see that everything cancels out to give zero, proving the claim.

Part (6) is *far* more tedious. We will thus defer this proof until later, and use it as an excuse to introduce one of the key ideas of the course. □

3. Multilinear functions

As you may have seen, a function of one variable is **linear** if we have both

$$f(x+y) = f(x) + f(y)$$

and

$$f(cx) = cf(x)$$

for any number c. You may have noticed that we made a big deal out of similar looking properties holding for both the dot and cross products. In your calculus class, you used these two types of properties for derivatives all the time, as without them calculus would be unbearable (and not worthy of the name "calculus"). In general, the following types of functions play an important role in linear algebra.

Definition. A function f of m vectors $v_1, \dots v_m$ is multilinear if it is linear in each argument individually. That is,

$$f(v_1, \dots v_{j-1}, v_j+v, v_{j+1}, \dots v_m) = f(v_1, \dots v_{j-1}, v_j, v_{j+1}, \dots v_m)+f(v_1, \dots v_{j-1}, v, v_{j+1}, \dots v_m)$$

and

$$f(v_1, \dots v_{j-1}, cv_j, v_{j+1}, \dots v_m) = cf(v_1, \dots v_{j-1}, v_j, v_{j+1}, \dots v_m)$$

for **all** vectors $v, v_1, \dots v_m$, all scalars $c \in \mathbb{R}$, and all indices $j = 1, \dots m$.

Returning to (6) in the theorem above, by the properties of $\cdot$ and $\times$ we can check (and its a good exercise for you!) that

$$f(u, v, w) = u \times (v \times w) - (u \cdot w)v + (u \cdot v)w$$

is a multilinear function of three three-dimensional vectors. The claim of (6) is equivalent to the claim that f is identically the zero function. This, of course, is infinitely many claims in one, but the multilinearity allows us to check that it is zero for **finitely** many choices and know that it holds for all choice. Why should this be true? Well, here is an example, considering f to be an arbitrary multilinear function of three three-dimensional vectors evaluated at a random point. We will express the answer in terms of the standard basis vectors i, j, k above.

$$\begin{aligned}
&f\left((1,2,0),(7,5,4),(3,0,0)\right) = f\left(i+2j, 7i+5j+4k, 3i\right) \\
&= f\left(i, 7i+5j+4k, 3i\right) + 2f\left(j, 7i+5j+4k, 3i\right) \\
&= 7f\left(i,i,3i\right) + 5f\left(i,j,3i\right) + 4f\left(i,k,3i\right) + 14f\left(j,i,3i\right) + 10f\left(j,j,3i\right) + 8f\left(j,k,3i\right) \\
&= 21f\left(i,i,i\right) + 15f\left(i,j,i\right) + 12f\left(i,k,i\right) + 42f\left(j,i,i\right) + 30f\left(j,j,i\right) + 24f\left(j,k,i\right).
\end{aligned}$$

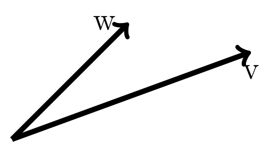
Playing around with this, or similar examples, you should convince yourself then that (6) holds if and only if it holds for all choices when u, v, w are one of the i, j, k. This is 27 choices left to check, but most of the choices are repeat calculations of other choices, so its not too bad to do.

You may also still have wondered about the nagging question of the geometric interpretation of cross products. The first hint in this direction follows directly if we plug in $u = v$ into (5) of the theorem above, which gives $v \cdot (v \times w) = -v \cdot (v \times w)$, meaning that its zero. By our result above, this means that v and $v \times w$ are orthogonal. Similarly, w and $v \times w$ are orthogonal. We record this for future reference.

Corollary. *For any v, w, we have*

$$v, w \perp v \times w.$$

This almost fixes the direction of $v \times w$. Namely, it pins it down to two choices. For example, in the following picture, v and w lie in the plane of this piece of paper (or screen), and so $v \times w$ either points directly out of or into the page:



To find out which choice it is, one can use the **right-hand rule**, which states that the direction is given by taking your **right** index finger, pointing it along v, and curling it towards w. Then the direction of your thumb is the direction of $v \times w$. In the picture above, $v \times w$ points out of the page, while $w \times v$ points into the page. Note that this is consistent with the above anti-commutativity property $v \times w = -w \times v$.

Finally, to determine $v \times w$ completely on a geometric level, we must also determine its magnitude. This is given as follows, which in fact derives from our geometric description of $v \cdot w$.

Theorem. *We have*

$$|v \times w| = |v||w| \sin \vartheta.$$

Proof. We make use of the following simple algebraic identity (try it for yourself!):

$$|v \times w|^2 = |v|^2|w|^2 - |v \cdot w|^2.$$

Now the right hand side can be simplified as

$$|v|^2|w|^2 - v|^2|w|^2 - \cos^2 \vartheta = |v|^2|w|^2(1 - \cos^2 \vartheta) = |v|^2|w|^2 \sin^2 \vartheta.$$

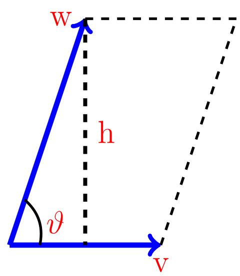
As $\sin \vartheta > 0$ (note that $0 \leq \vartheta < \pi$), and as lengths of vectors are positive numbers, we can take square roots to conclude the desired result. □

There is a useful geometric application, which can be used to find areas of parallelograms.

Corollary. *The parallelogram with vertices at the origin,* $(v_1, \ldots, v_3)$*,* (w_1, w_2, w_3)*, and* $(v_1 + w_1, v_2 + w_2, v_3 + w_3)$ *has area given by* $|v \times w|$*.*

Proof. Consulting the following diagram, recall that for any parallelogram, the area is the height times the length of a base, which in this case gives, by basic trigonometry,

$$A = |v|h = |v||w| \sin \vartheta = |v \times w|.$$



□

This gives a convenient, completely algebraic way to compute areas of parallelograms.