LECTURE 12: PROPERTIES OF VECTOR SPACES AND SUBSPACES

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1. Properties of vector spaces

Last time, we introduced the new notion of a vector space, an algebraic structure central to the theory of linear algebra. We saw a few examples of such objects. Right now, we want to build up some more theory about them. We begin with a few basic properties. Throughout, V will always denote a vector space

Lemma. *If V is a vector space and u, v, w are vectors such that $u + w = v + w$, then $u = v$.*

Proof. By the axioms defining vector spaces, there is an additive inverse x for w such that $w + x = 0$. Thus,

$$u = u + 0 = u + (w + x) = (u + w) + x = (v + w) + x = v + (w + x) = v + 0 = v.$$

□

Lemma. *If $v \in V$ is such that $v + u = u$ for all $u \in V$, then $v = 0$.*

Proof. In particular, there is a $0 \in V$ for all vector spaces V, and thus by assumption $v + 0 = 0 = 0 + 0$. By the last lemma, we can cancel the 0's on the right hand sides of both to get $v = 0$. □

Lemma. *For any $v \in V$, the additive inverse w for which $v + w = 0$ is unique.*

Proof. This follows by an identical proof as the proof of uniqueness of inverses of matrices. Namely, if $v+w = 0$, then $v+w = w+v = 0$ (since vector addition is commutative), and we suppose that there is another inverse $w' \in V$ with $v + w' = w' + v = 0$. Thus,

$$w = w + 0 = w + (v + w') = w + v + w' = (w + v) + w' = 0 + w' = w'.$$

□

A few other properties, whose proofs all follow quickly from the definitions, are given in the following result.

Lemma. *The following hold for any vector space V.*

(1) *The scalar product $0v = 0$ for all $v \in V$.*

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(2) *For any scalar $c \in F$, the scalar product with the 0 vector is 0, i.e., $c0 = 0$.*
(3) *If $c \in F$ and $v \in V$ with $cv = 0$, then $c = 0$ or $v = 0$.*

2. Subspaces

There is one particularly important type of vector space which will come up constantly for us.

Definition. A **subspace** of a vector space V is a subspace W if W is a vector space under the same vector addition and scalar multiplication operations as V.

Example. *The vector space V is always a subspace of V (a set is considered a subset of itself). If we want to avoid this situation, we can call a* **proper** *subspace any subspace W which is strictly smaller than V (i.e., there is some vector $v \in V$ which is not in W). At the other extreme, we always have the* **trivial subspace** *$\{0\}$, for which all the vector space axioms are, well, trivial.*

Example. *Subspaces of $\mathbb{R}^n$ are "flat" geometric objects* **passing through the origin**. *Note that this is strictly necessary, as* **all vector spaces must have a zero**, *and this zero has to be the zero of $\mathbb{R}^n$ (origin). So, for example, the subspaces of $\mathbb{R}^n$ are: $\mathbb{R}^n$ itself, planes passing through the origin, lines passing through the origin, and the singleton $\{0\}$.*

If we have a vector space V and a subset W, to check whether W is a subspace or not by checking all 10 vector space axioms is silly, even though this is the direct definition. Several of these axioms automatically hold; for example, all sums of two elements in V commute, then since W is a subset of V and the vector addition operation on a possible subspace is by definition the same as that for V, addition is automatically commutative on every subset of W. Subsets which aren't subspaces include the example above of any subset not containing 0, as well as sets which aren't closed under addition (for example the subset $\{i+j\} \subseteq \mathbb{R}^2$), and subsets not closed under scalar multiplication (for example, $\mathbb{Z} \subset \mathbb{R}$ is a subset which is closed under addition (the sum of two integers is an integer) and contains 0, but it isn't closed under scalar multiplication as π isn't an integer). It turns out that these are the only restrictions. Although it is not difficult, we omit the details of the following result which is frequently convenient.

Theorem. *If V is a vector space and W is a subset of V, then W is a subspace if and only if the following hold:*

(1) *$0 \in W$.*
(2) *$u + v \in W$ for all $u, v \in W$*
(3) *$cv \in W$ for all $c \in F$, $v \in W$.*

Example. *The set of solutions to the equation $Ax = 0$ for an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. This is called the* **nullspace** *or* **kernel** *of A, denoted $\ker(A)$. To check it is a subspace, note that (1) is satisfied as $A0 = 0$, so that $0 \in \ker(A)$, and if $x, y \in \ker(A)$, then $A(x + y) = Ax + Ay = 0 + 0 = 0$, so that $(x + y) \in \ker(A)$. Finally, if $x \in \ker(A)$, then $A(cx) = c(Ax) = c0 = 0$, so that (3) holds.*

Example. *It is easy to check that a line through the origin is a subspace of* $\mathbb{R}^2$*, say. For example, if the line is the set of points* $\{cv : c \in \mathbb{R}\}$ *for some non-zero vector* $v \in \mathbb{R}^2$ *(recall our earlier lecture about equations of lines and planes), then clearly* 0 *is in this set, it is closed under addition* $cv + c'v = (c + c')v$*, and it is closed under scalar multiplication as* $c'(cv) = (c'c)v$*.*

Example. *The set of even polynomials over* $\mathbb{R}$*, those for which* $f(x) = f(-x)$*, is a subspace of* $\mathcal{P}(\mathbb{R})$*. Indeed, denote the set of even polynomials by* $\mathcal{E}(\mathbb{R})$*. Then clearly* $0 \in \mathcal{E}(\mathbb{R})$*, if* f, g *are even polynomials then* $(f+g)(-x) = f(-x)+g(-x) = f(x)+g(x) = (f+g)(x)$ *so that* $f+g$ *is even, and if* f *is even and* $c \in \mathbb{R}$*, then* $(cf)(-x) = c(f(-x)) = cf(x) = (cf)(x)$*, so the polynomial* cf *is even too.*

Example. *The space of polynomials* $\mathcal{P}(\mathbb{R})$*, considered as functions of one real variable, is a subspace of the set of smooth functions on* $\mathbb{R}$*,* $\mathcal{C}^\infty$*, since it is a vector space under the same operations as* $\mathcal{C}^\infty$ *and it is clearly contained in it (i.e., every polynomial is smooth).*

Example. *The set of all* $n \times n$ *real matrices with non-zero entries isn't a subspace of* $M_{n\times n}(\mathbb{R})$ *as it doesn't contain the* 0 *matrix. For the same reason, the set of invertible* $n \times n$ *matrices is also not a subspace.*

Example. *The set of all upper triangular matrices (i.e., those with all zeros below the main diagonal) of size* $n \times n$ *with real entries is a subspace of* $M_{n\times n}(\mathbb{R})$ *is a subspace. Indeed, this clearly follows from basic matrix arithmetic. Another interesting subspace is given as follows. Define the* **trace** *of a matrix* $A \in M_{n\times n}(\mathbb{R})$*, denoted* $\mathrm{tr}(A)$*, as the sum of its diagonal entries, that is,*

$$\mathrm{tr}(A) = A_{11} + A_{22} + \ldots + A_{nn}.$$

Then the set of elements with trace 0 *is a subspace. Indeed, this is clear since* 0 *has trace* 0*, and since* $\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B)$*,* $c \cdot \mathrm{tr}(A) = \mathrm{tr}(cA)$ *for all matrices* A, B *and all* $c \in \mathbb{R}$*.*

Example. *Recall that* $\mathbb{F}_2$ *is the field with two elements. There are exactly* 5 *subspaces of* $\mathbb{F}_2^2$*. They are*

$$\mathbb{F}_2 = \{(0,0),(0,1),(1,0),(1,1)\}, \quad \{(0,0),(0,1)\}, \quad \{(0,0),(1,0)\}, \quad \{(0,0),(1,1)\}, 0 = \{(0,0)\}.$$

Its not a bad idea to check for yourself that these are indeed all subspaces!

Example. *A way to build new subspaces from old is to consider their intersection. Namely, if* V_i *are a bunch of vector spaces, then the intersection (set of common vectors belonging to all of them) is a subspace. For example, in* $\mathbb{R}^3$*, the intersection of two planes through the origin might be a line (through the origin of course), or it might be the whole plane, and the intersection of a line through the origin with a plane through the origin may be the trivial subspace* $\{0\}$ *or it might be the whole line itself.*

Example. *The union of vector spaces is not always a vector space. For example, the* x *and* y*-axes of* $\mathbb{R}^2$ *are subspace, but the union, namely the set of points on both lines, isn't a vector space as for example, the unit vectors* i, j *are in this union, but* $i + j$ *isn't.*

Example. *The set of all upper triangular* $n \times n$ *matrices with trace zero is a vector space, as it is the intersection of the subspaces of upper triangular matrices with the subspace of trace zero matrices in the vector space* $M_{n\times n}$.