LECTURE 11: 3×3 DETERMINANTS AND VECTOR SPACES

MA1111: LINEAR ALGEBRA I, MICHAELMAS 2016

1. 3×3 determinants

We have seen quite a lot about determinants, from their definitions to their applications to invertibility questions of matrices and in describing solutions of systems of equations, and concluding with some methods for computing them. It turns out that the Laplace expansion method is still pretty inefficient in general, but better methods (such as decomposing matrices into triangular matrices like those which appeared on the most recent HW) are studied in a more advanced linear algebra course.

However, there are two cases when we can just evaluate determinants directly. The first is the case of 2×2 matrices, which we saw can be evaluated by the simple formula (which you should just memorize):

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

For 3×3 matrices, there is another trick, called the **rule of Sarrus**, which you are free to assume for the rest of the course. Note that this is **only applicable in the** 3×3 **case**; in general similar diagrams you could draw for finding determinants won't work. Without proof, we can state the method as follows. Consider the 3×3 matrix

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix},$$

whose determinant we wish to compute, and copy over its left two columns and write them to the right of the matrix:

$$\begin{matrix} A_{11} & A_{12} & A_{13} & A_{11} & A_{12} \\ A_{21} & A_{22} & A_{23} & A_{21} & A_{22}. \\ A_{31} & A_{32} & A_{33} & A_{31} & A_{32} \end{matrix}$$

Now draw 6 diagonal slashes labelled with $\pm$ signs as follows:

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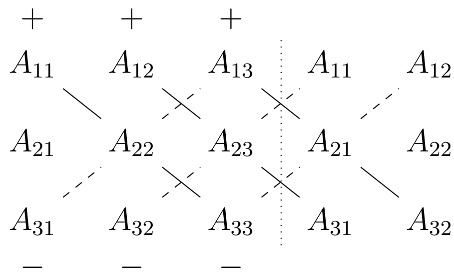


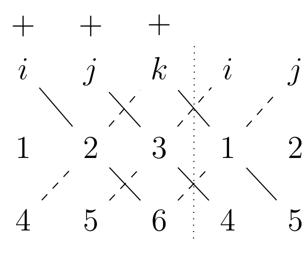
The determinant of A will then be the sum of the six products of terms along these diagonals weighted by the corresponding signs. That is,

$$\det A = A_{11}A_{22}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32} - A_{31}A_{22}A_{13} - A_{32}A_{23}A_{11} - A_{33}A_{21}A_{12}.$$

Finally, it is worthwhile to mention that the reason the definition of cross products looked strange before was that we didn't have determinants, which make the definition straightforward. Namely, if $v = (v_1, v_2, v_3)$, $w = (w_1, w_2, w_3)$, and if i, j, k are the usual unit basis vectors, then you can check that

$$v \times w = \det \begin{pmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}.$$

For example, $(1, 2, 3) \times (4, 5, 6) = \det \begin{pmatrix} i & j & k \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$, which is evaluated via the following diagram:



giving $(1, 2, 3) \times (4, 5, 6) = 12i + 12j + 5k - 8k - 15i - 6j = -5i + 7j - 3k = (-3, 6, -3)$. This is not only handy for computations, but also makes the definition much easier to remember and immediately explains some of the cross products' basic properties, such as anti-commutativity.

2. Vector spaces

We now switch gears entirely to discuss the main general object in linear algebra. When I told you that a vector was an "arrow" with direction and magnitude, this was somewhat of a lie, although I was a little closer to the truth when I said that a vector can be thought of as a tuple of numbers. It turns out that there is a general notion of vector, and that many things, including matrices, and even functions, can be vectors. The difficulty in telling you what a vector is is that in fact, it isn't possible to tell you the definition until I define a **vector space**, which is a set of a bunch of vectors! No, this definition isn't circular, its just that you have been trained up until this point to think of vectors first, and then in terms of the spaces (like $\mathbb{R}^n$) they live in, but a rigorous mathematical definition goes the other way (although of course the modern presentation is the opposite of how things were discovered). This abstraction isn't a useless gesture, over time mathematicians realized that the same patterns and arguments were being used over and over again, and this level of generality allows for one to apply the method to a much larger set of applications.

Definition. A vector space V over a field F (see below for a discussion of fields) is a set with two operations $+$ (vector addition) and $\cdot$ (scalar multiplication) such that the following properties hold:

(1) (Closure under addition) If $v, w \in V$, then $v + w \in V$.
(2) (Closure under scalar multiplication) If $c \in F$ and $v \in V$, then $cv \in V$.
(3) (Commutativity) For all $v, w \in V$, we have $v + w = w + v$.
(4) (Associativity of addition) For all $u, v, w \in V$, we have $(u+v)+w = u+(v+w)$.
(5) (Existence of 0) There is a vector $0 \in V$ for which $v + 0 = v$ for all $v \in V$.
(6) (Additive inverses) For all $v \in V$, there is a vector $w \in W$ for which $v + w = 0$.
(7) (Associativity of scalar multiplication) If $\alpha, \beta \in F$ and $v \in V$, then $(\alpha\beta)v = \alpha(\beta v)$.
(8) (Distributivity across vector addition) If $c \in F$ and $v, w \in V$, then $c(v + w) = cv + cw$.
(9) (Distributivity across vector addition) If $\alpha, \beta \in F$ and $v \in v$, then $(\alpha + \beta)v = \alpha v + \beta v$.
(10) (Multiplication by 1) If 1 is the multiplicative identity of the field F, then for all $v \in V$, $1v = v$.

As you may have noticed, elements $v \in V$ are called vectors and elements $c \in F$ are called scalars. As for what a field is, the short answer is that it is something like a set of numbers (which distinguished elements called 0 and 1 which act like 0 and 1 do in the real numbers) which have a multiplication operation and an addition operation, which satisfies "all the usual properties" of real number multiplication and addition, and in which you can "divide" by any non-zero element. These are the subject of a proper course in abstract algebra all on their own, so for now, we will only note that there are a few examples to keep in mind. In general, "sets with extra structure" (and maps between them) constitutes the discipline of algebra.

For example, the set of real numbers $\mathbb{R}$ **is** a field, as are the sets $\mathbb{Q}$ of rational numbers (fractions), and the set of complex numbers $\mathbb{C}$. However, the set $\mathbb{Z}$ of integers is **not** a field, as multiplicative inverses don't exist **within the field** (i.e., you can divide 1 by 2 to get 1/2, but this isn't an integer). Another very important example is the following.

Example. *There is a field $\mathbb{F}_2$ with two elements. As a set,*

$$\mathbb{F}_2 = \{0, 1\}.$$

Multiplication and addition work by multiplying zero and one as real numbers and reducing them "modulo 2", which states that an integer is 0 modulo 2 if its even and 1 modulo 2 if its odd. For example, in this field, $1 + 1 = 0$. This is a common field to use in computer science, and things involving this field are often called "Boolean". You can think of the formula $1 + 1 = 0$ as encoding something like a transistor, where a current is either off or on and adding 1 toggles the state of the transistor.

Finally, here is one more example of a field.

Example. *The set*

$$\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{R}\}$$

is a field, with the multiplication and addition working on the elements of $\mathbb{Q}(\sqrt{2})$ usual real numbers. Part of the definition of a field states that the field must be closed under multiplication, which can be seen directly by multiplying:

$$(a + b\sqrt{2})(c + d\sqrt{2}) = (ac + 2db) + (b + d)\sqrt{2} \in \mathbb{Q}(\sqrt{2}).$$

For not surprising reasons, this is called a number field, and the study of such sets of numbers built out of the rational numbers by "adding in" roots of polynomial equations is a major topic called algebraic number theory.

Finally, we come to a few examples of vector spaces.

Example. *If F is a field and n is a positive integer, then the set of all n-tuples of elements in F,*

$$F^n = \{(c_1, c_2 \dots, c_n) : c_1, c_2, \dots, c_n \in F\},$$

is a vector space when equipped with the addition rule

$$(c_1, c_2 \dots, c_n) + (c_1', c_2' \dots, c_n') = (c_1 + c_1', c_2 + c_2' \dots, c_n + c_n')$$

and the scalar multiplication rule

$$c(c_1, c_2 \dots, c_n) = (cc_1, cc_2 \dots, cc_n).$$

When $F = \mathbb{R}$, this is the vector space $\mathbb{R}^n$, the set of all n-dimensional vectors with real entries.

Example. *The set of all $m \times n$ matrices with entries in F is a vector space over F, with matrix addition and scalar multiplication of matrices working in the usual way. We denote this field by $M_{m\times n}(F)$. In the most common case of $F = \mathbb{R}$, we also write $M_{m\times n} = M_{m\times n}(\mathbb{R})$.*

Example. *The set $\mathcal{C}^\infty$ of smooth (infinitely differentiable functions) of one real variable is a vector spaceover $\mathbb{R}$, under the usual definitions of function addition and multiplication of a function by a constant (i.e., defined "pointwise"). This follows from the fact that the derivative operator $\partial/\partial x$ is linear, that is $(f+g)' = f' + g'$ and $(cf') = c(f')$, where f and g are functions and $c \in \mathbb{R}$. This will turn out to be an* **infinite dimensional** *vector space, since we cannot find a finite set of functions which describes all smooth functions (in contract to the case of $\mathbb{R}^n$, where all vectors are "built" out of standard unit vectors $e_1, \ldots, e_n$).*

Example. *The set of all* **polynomials** *over F is a vector space. The set is given by*

$$\mathcal{P}(F) = \{f(x) = a_n x^n + \ldots + a_0 : a_n, \ldots \in F\},$$

polynomial addition is given by addition of coefficients:

$$(a_n x^n + \ldots + a_0) + (b_n x^n + \ldots + b_0) = (a_n + b_n)x^n + \ldots + (a_0 + b_0),$$

and scalar multiplication is given by multiplying each coefficient by a scalar:

$$c(a_n x^n + \ldots + a_0) = (ca_n)x^n + \ldots + (ca_0).$$