TUTORIAL 2

MA1111: LINEAR ALGEBRA I, MICHAELMAS 2016

(1) Suppose A is a 3×3 matrix whose third row is a linear combination of the first two rows. Show that A is not invertible and find a vector b such that $Ax = b$ has no solutions. Find a vector b for which it has infinitely many solutions.

Solution:

Suppose that the third row r_3 is a linear combination $r_3 = \alpha r_1 + \beta r_2$. Then our matrix has the shape

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \alpha a_{11} + \beta a_{21} & \alpha a_{12} + \beta a_{22} & \alpha a_{13} + \beta a_{23} \end{pmatrix}.$$

This has determinant 0, as subtracting α times the first row and β times the second row gives a row of zeros, but preserves the determinant, and hence the matrix isn't invertible. Now suppose our vector is $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3. \end{pmatrix}$. Then using row-reduction, we get a row of zeros left of the bar | by subtracting α times the first row and β times the second row from the third row:

$$\left(\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ \alpha a_{11} + \beta a_{21} & \alpha a_{12} + \beta a_{22} & \alpha a_{13} + \beta a_{23} & b_3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ 0 & 0 & 0 & b_3 - \alpha b_1 - \beta b_2 \end{array}\right).$$

This will have no solutions if $b_3 - \alpha b_1 - \beta b_2 \neq 0$. Otherwise, if there are no inconsistencies in the first and second rows, which we can guarantee by letting $b_1 = b_2 = 0$, then there will be at least one free variable and hence infinitely many solutions whenever $b_3 - \alpha b_1 - \beta b_2 = 0$. To find a vector with $b_3 - \alpha b_1 - \beta b_2 \neq 0$, just pick $b_1 = 0$, $b_2 = 0$, $b_3 = 1$. In the second case, just pick $b_1 = b_2 = b_3 = 0$.

In other words, we have found that $b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ yields a system with no solutions, and the vector $b = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ yields a system with infinitely many solutions.

(2) Using cofactors, find the determinant and inverse of the matrix

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 5 & 1 & 4 \\ 0 & 1 & 0 \end{pmatrix}.$$

Solution:
We first compute the matrix of minors of A as

$$M = \begin{pmatrix} \det\begin{pmatrix} 1 & 4 \\ 1 & 0 \end{pmatrix} & \det\begin{pmatrix} 5 & 4 \\ 0 & 0 \end{pmatrix} & \det\begin{pmatrix} 5 & 1 \\ 0 & 1 \end{pmatrix} \\ \det\begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix} & \det\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} & \det\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \\ \det\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} & \det\begin{pmatrix} 1 & 2 \\ 5 & 4 \end{pmatrix} & \det\begin{pmatrix} 1 & 3 \\ 5 & 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} -4 & 0 & 5 \\ -2 & 0 & 1 \\ 10 & -6 & -14 \end{pmatrix}.$$

By inserting signs into the minor matrix, we get the matrix of cofactors:

$$C = \begin{pmatrix} -4 & 0 & 5 \\ 2 & 0 & -1 \\ 10 & 6 & -14 \end{pmatrix}.$$

By expanding $\det A$ along the bottom row, we find that the determinant is equal to $1 \cdot C^{32} = 6$. By taking the transpose of the cofactor matrix, we get the adjugate

$$\text{adj}(A) = \begin{pmatrix} -4 & 2 & 10 \\ 0 & 0 & 6 \\ 5 & -1 & -14 \end{pmatrix}.$$

Now the determinant is non-zero, and so the matrix is invertible with inverse $A^{-1} = (\det A)^{-1}\text{adj}A$, which we plug in to find is

$$A^{-1} = \begin{pmatrix} -\frac{2}{3} & \frac{1}{3} & \frac{5}{3} \\ 0 & 0 & 1 \\ \frac{5}{6} & -\frac{1}{6} & -\frac{7}{3} \end{pmatrix}.$$

(3) A very famous puzzle which was all the rage in the 19th century is the famous *15 puzzle*, which you have likely seen some version of. The puzzle asks you to solve the following problem. Suppose we have a 4×4 grid of sliding pieces, with 15 moving pieces and one empty square, arranged as follows:

1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	

That is, the numbers are all in order except that 14 and 15 are flipped, and the lower right hand corner is a blank space. Pieces can be shuffled around so that there is always one blank space; for example, one could shift the 12 down in the above configuration to give the arrangement

1	2	3	4
5	6	7	8
9	10	11	
13	15	14	12

.

The question is, can we shuffle around the pieces to make the puzzle pieces all line up in order, i.e., switch the 14 and the 15? The answer is no, and we can use permutations to see why. For any configuration, we can define an *invariant* as follows. Label the blank space by 16 and read define an associated permutation which assigns the sequence $1, 2, \ldots, 16$ the sequence of numbers which reads off the rows from left to right and then from top to bottom. For example, the first configuration above corresponds to the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 15 & 14 & 16 \end{pmatrix},$$

while the second configuration corresponds to

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 16 & 13 & 15 & 14 & 12 \end{pmatrix}.$$

Define the *taxicab* distance as the sum number of rows plus the number of columns that the blank space is away from the lower right corner, i.e., $0+0=0$ in the first case and $1+0=1$ in the second case. Let's say that the parity of a permutation is 0 if its even, and 1 if its odd, and similarly an integer's parity is 0 if its even and 1 if its odd. Define the *t-invariant* of a puzzle configuration as 0 if the parity of the permutation and the taxicab numbers are the same, and 1 otherwise. Note that if I make a move on the puzzle grid, I move the blank space by one, and so change the taxicab distance by exactly one, and so flip the parity of the taxi distance, and on the side of permutations, I am composing the original permutation with a transposition, and hence change the sign of the permutation as well. Thus, the t-invariant is always fixed for a given puzzle, (unless you break it into pieces with a hammer).

Now, the original 15 puzzle, where 14 and 15 are flipped, isn't solvable, as the permutation is just the transposition $(14\ 15)$, which has sign -1, or as we said above parity 1, and the taxicab number is even (so has parity 0). Hence, the t-invariant is 1, while the permutation corresponding to the numbers in order is just the permutation with no transpositions (or, if you like, just write it as something like $(12)(12)$), and so has sign $+1$ or parity 0, while the taxicab number is still 0, and so the t-invariant is 0. These two numbers have different parities, implying the puzzle is impossible to solve.

Now, here is your question. Is the puzzle with the following configuration solvable, that is, can you shift around puzzle pieces to get the numbers 1 through 15 back in order:

3	6	4	9
7	5	2	8
12	10		1
15	14	13	11

Solution:

This puzzle corresponds to the permutation

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 3 & 6 & 4 & 9 & 7 & 5 & 2 & 8 & 12 & 10 & 16 & 1 & 15 & 14 & 13 & 11 \end{pmatrix}.$$

We find the parity of π by first writing down its cycle decomposition:

$$\pi = (1\ 3\ 4\ 9\ 12)(2\ 6\ 5\ 7)(11\ 16)(13\ 15).$$

We write each cycle as a product of transpositions to get

$$\pi = (1\ 12)(1\ 9)(1\ 4)(1\ 3)(2\ 7)(2\ 5)(2\ 6)(11\ 16)(13\ 15).$$

There are 9 transpositions here, and so the sign of π is -1 and the parity is 1. The taxicab number is $1+1=2$, which has parity 0. Thus, the t-invariant is 1. As the t-invariant is different than that t-invariant of the "solved" position with the numbers in order, this puzzle can never be solved.

Advanced Problem (optional): Show that if A, B are square matrices with $A+B = AB$, then $AB = BA$. (Hint: on the homework, you will show that if $AB = I_n$, then $B = A^{-1}$).

Solution:

We rewrite the equation $A + B = AB$ as

$$I_n + A + B = AB + I_n,$$

or

$$I_n = AB - A - B - I_n = (A - I_n)(B - I_n).$$

Thus, using the hint, we have

$$I_n = (B - I_n)(A - I_n),$$

which upon expanding becomes

$$I_n = BA - A - B - I_n,$$

so that

$$A + B = BA.$$

But $A + B = AB$ by assumption, so $AB = BA$.