TUTORIAL 1, SOLUTIONS

MA1111: LINEAR ALGEBRA I, MICHAELMAS 2016

i). In class, we mentioned that the cross product is not associative. That is, we don't always have $u \times (v \times w) = (u \times v) \times w$. Instead, the cross product satisfies an important identity known as the *Jacobi identity*:

$$u \times (v \times w) + v \times (w \times u) + w \times (u \times v) = 0. \tag{1}$$

Show, using the identity

$$u \times (v \times w) = (u \cdot w) \cdot v - (u \cdot v) \cdot w$$

which we showed in class, that (1) holds for any vectors $u, v, w \in \mathbb{R}^3$.

Solution:

We plug our identity into the left hand side of (1), yielding:

$$\begin{aligned}
&u \times (v \times w) + v \times (w \times u) + w \times (u \times v) \\
&= (u \cdot w) \cdot v - (u \cdot v) \cdot w + (v \cdot u) \cdot w - (v \cdot w) \cdot u + (w \cdot v) \cdot u - (w \cdot u) \cdot v \\
&= 0,
\end{aligned}$$

where in the last line we used the fact that $a \cdot b = b \cdot a$ for all vectors a, b.

ii). Prove that the diagonals of a square are orthogonal.

Solution: We may assume that the square has lower left vertex at the origin $A = (0, 0)$, with its other three vertices at $B = (a, 0)$, $C = (0, a)$, and $D = (a, a)$, where of course a is the side length of the square. The two diagonals are parallel to the vectors

$$\overrightarrow{AD} = (a, a)$$

and

$$\overrightarrow{BC} = (-a, a).$$

Thus, we just need to show that $\overrightarrow{AD}$ is orthogonal to $\overrightarrow{BC}$, which we have shown in class is equivalent to their dot product being zero. This is easily verified:

$$\overrightarrow{AD} \cdot \overrightarrow{BC} = -a^2 + a^2 = 0.$$

iii). Consider the three planes given by

$$x + 2y + z = 5, \qquad 2x + 2y + 2z = 4, \qquad x + z = -1.$$

Using row reduction, find the intersection between all three planes.

Solution: We represent the intersection as the set of solutions to the system of equations

$$\begin{cases} x + 2y + z = 5 \\ 2x + 2y + 2z = 4 \\ x + z = -1, \end{cases}$$

which is represented by the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 2 & 2 & 2 & 4 \\ 1 & 0 & 1 & -1 \end{array}\right).$$

We row reduce as follows:

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 2 & 2 & 2 & 4 \\ 1 & 0 & 1 & -1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 0 & -2 & 0 & -6 \\ 0 & -2 & 0 & -6 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & -2 & 0 & -6 \\ 0 & 0 & 0 & -0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & -0 \end{array}\right).$$

Note that there are two pivotal variables (x, y) and one free one (z). We can then read off the solution set by letting $z = t$ for any $t \in \mathbb{R}$ and then

$$x + z = x + t = -1 \implies x = -1 - t,$$

$$y = 3.$$

Thus, the three planes intersect in a line with parametric equations

$$\begin{cases} x = -1 - t \\ y = 3 \\ z = t. \end{cases}$$

iv). Show that if $ad - bc \neq 0$, then the reduced row echelon form of the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Solution: We have to consider two cases. If $a \neq 0$, then we row reduce as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} 1 & \frac{b}{a} \\ c & d \end{pmatrix} \to \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & d - \frac{bc}{a} \end{pmatrix} = \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & \frac{ad-bc}{a} \end{pmatrix}.$$

Now, as $ad - bc$ is not zero, we can divide by it and can continue reducing to

$$\begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Finally, if $a = 0$, then since $ad - bc \neq 0$ we also have $c \neq 0$ and so we may reduce as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & b \\ c & d \end{pmatrix} \to \begin{pmatrix} c & d \\ 0 & b \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & \frac{d}{c} \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Advanced Problem (optional):

For any three numbers x, y, z define the three vectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad v_3 = \begin{pmatrix} x^2 \\ y^2 \\ z^2 \end{pmatrix}.$$

For which choices of x, y, z is the set of all linear combinations of the three vectors, $\mathrm{span}(v_1, v_2, v_3)$, equal to all of $\mathbb{R}^3$?

Solution: We put the vectors in a matrix and row reduce:

$$\begin{pmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{pmatrix}.$$

Now, if $x \neq y$ and $x \neq z$, then we can divide the second and third rows giving

$$\begin{pmatrix} 1 & x & x^2 \\ 0 & 1 & x+y \\ 0 & 1 & x+z \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & x & x^2 \\ 0 & 1 & x+y \\ 0 & 0 & z-y \end{pmatrix}.$$

Now, if $y \neq z$, then we can finish row reducing to give

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now this matrix, for any augmented version with another column on the right, corresponds to a consistent system of equations. Thus, under our assumptions $x \neq y, x \neq z, y \neq z$, or more simply x, y, z are distinct, the vectors span all of $\mathbb{R}^3$. Conversely, if they aren't all distinct, by symmetry we can assume that $x = y$. But then the matrix reduces as

$$\begin{pmatrix} 1 & x & x^2 \\ 1 & x & x^2 \\ 1 & z & z^2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & x & x^2 \\ 0 & 0 & 0 \\ 0 & z-x & z^2-x^2 \end{pmatrix},$$

which will yield an inconsistent system whenever the column of the related augmented matrix doesn't have a 0 in the second row. In other words, any vector $(a, b, c) \in \mathbb{R}^3$ for which $a \neq b$ will not be in the span of v_1, v_2, v_3. This means that the span is confined to a plane (and could be just a line depending on whether $x = z$ or not).

Here is a simple example. If $x = 2, y = 2, z = 2$, then we obtain the vectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} 4 \\ 4 \\ 9 \end{pmatrix}.$$

One can check that $-6v_1 + 5v_2 = v_3$, which shows that v_3 is a linear combination of v_1 and v_2. Thus, adding in v_3 doesn't increase the span of the first two vectors (check why this is true!), and so the span of all three vectors is the same as the span of just v_1, v_2, which is just a plane.