HOMEWORK 7

MA1111: LINEAR ALGEBRA I, MICHAELMAS 2016

Solutions are due at the beginning of class on **Thursday, November 24**. Please put your name and course on your assignment, and make sure to staple your papers.

(1) Consider the basis $\{(-2,3,1),(3,-1,1),(1,-1,-1)\}$ of $\mathbb{R}^3$. Compute the coordinates of $v=(6,-2,1)$ with respect to this basis.

Solution: Consider the matrix A which has these basis vectors as its columns:

$$A = \begin{pmatrix} -2 & 3 & 1 \\ 3 & -1 & -1 \\ 1 & 1 & -1 \end{pmatrix}.$$

This matrix has determinant 6, which in particular verifies that these three vectors really were a basis of $\mathbb{R}^3$. To compute the coordinate vector, we first compute the inverse:

$$A^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{6} \\ \frac{2}{3} & \frac{5}{6} & -\frac{7}{6} \end{pmatrix}.$$

The product

$$A^{-1}v = \begin{pmatrix} \frac{1}{3} \\ \frac{11}{6} \\ \frac{7}{6} \end{pmatrix}$$

gives the desired coordinate vector.

(2) Consider the function $T\colon M_{2\times 3}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ given by

$$T\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} 2a_{11}-a_{12} & a_{13}+2a_{12} \\ 0 & 0 \end{pmatrix}.$$

(a) Show that T is a linear transformation.
(b) Find a basis for $\ker(T)$.
(c) Find a basis for $\mathrm{Im}(T)$.
(d) What does the rank-nullity theorem claim in this case? Check that this indeed holds, using your answers from (b) and (c).

Solution: (a): This can be directly checked:

$$\begin{aligned}
&T\left(\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23}\end{pmatrix} + \begin{pmatrix} a'_{11} & a'_{12} & a'_{13} \\ a'_{21} & a'_{22} & a'_{23}\end{pmatrix}\right) \\
&= \begin{pmatrix} 2(a_{11}+a'_{11}) - (a_{12}+a'_{12}) & (a_{13}+a'_{13}) + 2(a_{12}+a'_{12}) \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 2a_{11} - a_{12} & a_{13}+2a_{12} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 2a'_{11} - a'_{12} & a'_{13}+2'a_{12} \\ 0 & 0 \end{pmatrix} \\
&= T\left(\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23}\end{pmatrix}\right) + T\left(\begin{pmatrix} a'_{11} & a'_{12} & a'_{13} \\ a'_{21} & a'22 & a'_{23}\end{pmatrix}\right),
\end{aligned}$$

and

$$T\left(c\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23}\end{pmatrix}\right) = \begin{pmatrix} 2ca_{11} - ca_{12} & ca_{13}+2ca_{12} \\ 0 & 0 \end{pmatrix} = cT\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23}\end{pmatrix}.$$

(b): A matrix is sent to the zero matrix in $M_{2\times 2}(\mathbb{R})$ exactly if $2a_{11} = a_{12}$ and $a_{13} = -2a_{12}$ and $a_{13} = -2a_{12}$. Another way of putting this is that $a_{12} = 2a_{11}$ and $a_{13} = -4a_{11}$.Other than these conditions, there are no restrictions. Thus, a generic matrix in the kernel is a matrix of the shape

$$\begin{pmatrix} a_{11} & 2a_{11} & -4a_{11} \\ a_{21} & a_{22} & a_{23}\end{pmatrix} = a_{11}\begin{pmatrix} 1 & 2 & -4 \\ 0 & 0 & 0\end{pmatrix} + a_{21}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0\end{pmatrix} + a_{22}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0\end{pmatrix} + a_{23}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1\end{pmatrix}.$$

The four matrices on the right hand side of this equation thus span the kernel, and they are clearly linearly independent, so they form a basis of $\ker(T)$. In particular, the nullity of T, $\text{null}(T)$, is equal to 4.

(c):

Clearly, as a_{11}, a_{12}, a_{13} range over $\mathbb{R}$, so do $2a_{11} - a_{12}$ and $a_{13} + 2a_{12}$. For example, if we want to get the first one to be an arbitrary real number x and the second to be an arbitrary real number $\mathbb{R}$, we can simply choose $a_{11} = x/2$, $a_{12} = 0$, and $a_{13} = y$. Thus the set of matrices in the image of T, that is, those of the shape

$$\begin{pmatrix} 2a_{11} - a_{12} & a_{13}+2a_{12} \\ 0 & 0 \end{pmatrix},$$

are exactly the set of 2×2 matrices with a bottom row of zeros. In particular, a basis for $\text{Im}(T)$ is

$$\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0\end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}\right\}.$$

In particular, the rank of T is 2.

(d): The rank-nullity theorem states that the rank plus the nullity, i.e., $2 + 4$ equals the dimension of the "input" vector space. Indeed, the dimension of $M_{2\times 3}(\mathbb{R})$ is 6.

(3) Given linear transformations $T_1 \colon V \to W$ and $T_2 \colon W \to W'$ for vector spaces V, W, W', their composition $T = T_2 T_1 \colon V \to W'$ is their composition as functions. That is, if $v \in V$, then $T(v) = T_2(T_1(v)) \in W'$. Show that the composition T is also a linear transformation.

Solution:

Using the linearity of both T_1 and T_2, we directly compute that

$$T(v+v') = T_2(T_1(v+v')) = T_2(T_1(v) + T_1(v')) = T_2(T_1(v)) + T_2(T_1(v')) = T(v) + T(v')$$

and

$$T(cv) = T_2(T_1(cv)) = T_2(cT_1(v)) = cT_2(T_1(v)) = cT(v).$$

(4) We have seen that the subset of matrices in $M_{n\times n}(\mathbb{R})$ with trace zero (i.e., the sum of elements on their diagonals are zero) are a subspace of $M_{n\times n}(\mathbb{R})$. One way to find the dimension, as you did in a specific case on the last homework, is to explicitly write down a basis. However, there is another method, which this problem will guide you through.

(a) Show that the function $\mathrm{tr} \colon M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ which takes the trace of a matrix is a linear transformation.
(b) Describe the kernel and image of this transformation.
(c) Find the dimension of $\mathrm{Im}(\mathrm{tr})$.
(d) Using the rank-nullity theorem, find the dimension of the subspace of trace zero matrices in $M_{n\times n}(\mathbb{R})$.

Solution:

(a): This follows by the basic properties for trace which we have seen before. That is, for any matrices $A, B \in M_{n\times n}(\mathbb{R})$, we have

$$\mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B),$$
$$\mathrm{tr}(cA) = c\,\mathrm{tr}(A).$$

(b): The kernel of this map is the subspace of trace zero matrices, which is the space we are trying to compute the dimension of. The image is clearly $\mathbb{R}$, as we can find a matrix with an arbitrary trace α by taking the matrix with α in the upper-left corner and 0 everywhere else.

c): The dimension of the image is the dimension of $\mathbb{R}$, namely 1.

d): The rank-nullity theorem tells us that the dimension of the space we are looking for (which is the kernel of tr) plus the dimension of the image of tr (which we computed to be 1) is equal to the dimension of $M_{n\times n}(\mathbb{R})$, which we know to be n^2. Thus the dimension of the subspace of trace zero matrices is $n^2 - 1$.