HOMEWORK 6

MA1111: LINEAR ALGEBRA I, MICHAELMAS 2016

Solutions are due at the beginning of class on **Thursday, November 17**. Please put your name and course on your assignment, and make sure to staple your papers.

(1) Determine which of the following sets of vectors are linearly independent in $\mathbb{R}^4$.

(a)

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 4 \\ -1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 1 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} 3 \\ 4 \\ -4 \\ 3 \end{pmatrix}.$$

(b)

$$v_1 = \begin{pmatrix} -1 \\ 2 \\ 3 \\ 5 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 7 \\ 100 \\ -10 \\ 11 \end{pmatrix}, \quad v_3 = \begin{pmatrix} -\pi \\ 0 \\ 0 \\ 10 \end{pmatrix}, \quad v_4 = \begin{pmatrix} e \\ 0 \\ \pi^e \\ 11 \end{pmatrix}, \quad v_5 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

(c)

$$v_1 = \begin{pmatrix} 0 \\ -2 \\ 1 \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} 0 \\ 2 \\ 0 \\ -1 \end{pmatrix}, \qquad v_4 = \begin{pmatrix} 0 \\ 3 \\ 0 \\ 0 \end{pmatrix}.$$

Solution: a). The corresponding matrix with these vectors as its columns is

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 4 & 0 & -4 \\ -1 & 1 & 3 \end{pmatrix},$$

which has RREF

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

As this has only 2 pivots and there are three columns, the set $\{v_1, v_2, v_3\}$ is not linearly independent.

b). This set is not linearly independent as there are 5 vectors in $\mathbb{R}^4$, which must always be linearly dependent. This is because the matrix with these vectors as its columns has 4 rows but 5 columns and hence has at most 4 pivots, and cannot have a pivot in each column.

c). If we put the vectors in a matrix and take the determinant, we find that

$$\det \begin{pmatrix} 0 & 1 & 0 & 0 \\ -2 & 1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & -1 & 0 \end{pmatrix} = -\det \begin{pmatrix} -2 & 2 & 3 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix} = \det \begin{pmatrix} 2 & 3 \\ -1 & 0 \end{pmatrix} = 3 \neq 0.$$

Hence the four vectors are linearly independent (and in fact form a basis of $\mathbb{R}^4$).

(2) Find a basis of the space of 2×2 real matrices with trace zero, i.e., those matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$ with $a + d = 0$.

Solution: We claim that

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$$

is such a basis. Indeed, to show its linearly independent, suppose that there is a linear combination

$$\alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \beta \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} = 0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then clearly $\alpha = \beta = \gamma = 0$, so this is a trivial linear dependency. To show that they span the space of trace zero matrices, suppose that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has trace zero, i.e., $a + d = 0$ or $d = -a$. Then clearly

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

so that any matrix in our space is in the span of these three matrices. Hence, they form a basis of our space.

(3) Suppose that $\{v_1, \ldots v_k\}$ is a set of vectors in $\mathbb{R}^n$ and A is a matrix $A \in M_{m\times n}$. Further suppose that $\{Av_1, \ldots Av_k\}$ is a linearly independent set of vectors in $\mathbb{R}^m$. Show that the original set of vectors $\{v_1, \ldots, v_k\}$ is a linearly independent set of vectors in $\mathbb{R}^n$.

Solution:

Suppose that they are linearly dependent. That is, suppose that there are $c_1, \ldots, c_n \in \mathbb{R}$, not all equal to zero, for which

$$c_1 v_1 + \ldots + c_k v_k = 0.$$

Multiplying both sides of this equation on the left by A and using the basic properties of matrix arithmetic, we find that

$$c_1 A v_1 + \dots c_k A v_k = 0,$$

which shows that $\{Av_1, \dots Av_n\}$ is linearly dependent. This contradicts our assumptions about the latter set, and so the vectors $\{v_1, \dots, v_n\}$ are linearly independent.

(4) Find bases for the kernel (null space), row space, and column space of the matrix

$$A = \begin{pmatrix} 0 & 2 & 8 & -7 \\ 2 & -2 & 4 & 0 \\ -3 & 4 & -2 & 5 \end{pmatrix}.$$

Solution:

All three are found using the RREF of A, which is

$$\begin{pmatrix} 1 & 0 & 6 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The set of solutions in the kernel are those those of the form

$$\begin{cases} x_1 = -6t \\ x_2 = -4t \\ x_3 = t \\ x_4 = 0, \end{cases}$$

which is a line with basis $\{(-6, -4, 1, 0)\}$. The row space has as a basis the non-zero rows in the RREF of A, namely $\{(1, 0, 6, 0), (0, 1, 4, 0), (0, 0, 0, 1)\}$. The column space has as a basis the columns of A corresponding to the three columns with a pivot in the corresponding column of the RREF of A. That is, it has as a basis $\{(0, 2, -3), (2, -2, 4), (-7, 0, 5)\}$. In fact, the matrix with these three vectors as columns is invertible (check the determinant to be non-zero, for example), and so another basis of the column space is simply $\{e_1, e_2, e_3\}$. We could have also found this by taking the transpose of A and row reducing to find the row space of A^T.