HOMEWORK 4

MA1111: LINEAR ALGEBRA I, MICHAELMAS 2016

Solutions are due at the beginning of class on **Thursday, October 27**. Please put your name and student number, as well as your subject (Maths., TP, or TSM) on the back of your assignment, and make sure to staple your papers.

(1) Find the determinant of the matrix

$$A = \begin{pmatrix} 3 & 2 & 0 & 1 \\ 4 & 0 & 1 & 2 \\ 3 & 0 & 2 & 1 \\ 9 & 2 & 3 & 1 \end{pmatrix}.$$

Solution: We expand along the second column (since it has the most zeros):

$$\det A = -2 \det \begin{pmatrix} 4 & 1 & 2 \\ 3 & 2 & 1 \\ 9 & 3 & 1 \end{pmatrix} + 2 \det \begin{pmatrix} 3 & 0 & 1 \\ 4 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix}.$$

In the first 3×3 matrix, we expand along the second row (as it has the smallest entries) to find that

$$\det \begin{pmatrix} 4 & 1 & 2 \\ 3 & 2 & 1 \\ 9 & 3 & 1 \end{pmatrix} = -3 \det \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} + 2 \det \begin{pmatrix} 4 & 2 \\ 9 & 1 \end{pmatrix} - \det \begin{pmatrix} 4 & 1 \\ 9 & 3 \end{pmatrix}$$
$$= -3(-5) + 2(-14) - (3) = 15 - 28 - 3 = -16.$$

On the other 3×3 matrix, we expand along the second column to find:

$$\det \begin{pmatrix} 3 & 0 & 1 \\ 4 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix} = \det \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} - 2 \det \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$$
$$= 0 - 2(2) = -4.$$

Overall, we find that $\det A = 24$.

(2) Use the method of adjoints to compute the inverse of the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 7 & 0 & 9 \end{pmatrix}.$$

Solution: We first compute the minor matrix of A to be

$$\begin{pmatrix} \det\begin{pmatrix}5 & 0\\ 0 & 9\end{pmatrix} & \det\begin{pmatrix}0 & 0\\ 7 & 9\end{pmatrix} & \det\begin{pmatrix}0 & 5\\ 7 & 0\end{pmatrix} \\ \det\begin{pmatrix}2 & 3\\ 0 & 9\end{pmatrix} & \det\begin{pmatrix}1 & 3\\ 7 & 9\end{pmatrix} & \det\begin{pmatrix}1 & 2\\ 7 & 0\end{pmatrix} \\ \det\begin{pmatrix}2 & 3\\ 5 & 0\end{pmatrix} & \det\begin{pmatrix}1 & 3\\ 0 & 0\end{pmatrix} & \det\begin{pmatrix}1 & 2\\ 0 & 5\end{pmatrix} \end{pmatrix} = \begin{pmatrix} 45 & 0 & -35 \\ 18 & -12 & -14 \\ -15 & 0 & 5 \end{pmatrix}.$$

By inserting signs, we find the cofactor matrix to be

$$\begin{pmatrix} 45 & 0 & -35 \\ -18 & -12 & 14 \\ -15 & 0 & 5 \end{pmatrix}.$$

By taking the transpose, we find the adjoint matrix to be

$$\operatorname{adj}(A) = \begin{pmatrix} 45 & -18 & -15 \\ 0 & -12 & 0 \\ -35 & 14 & 5 \end{pmatrix}.$$

On the last homework, we found $\det A = -60$, and dividing the adjoint by this yields the inverse we are after:

$$A^{-1} = \begin{pmatrix} -\frac{3}{4} & \frac{3}{10} & \frac{1}{4} \\ 0 & \frac{1}{5} & 0 \\ \frac{7}{12} & -\frac{7}{30} & -\frac{1}{12} \end{pmatrix}.$$

(3) A matrix A is called *upper triangular* if all entries below the main diagonal are 0, i.e., if $A_{ij} = 0$ whenever $j < i$. For example,

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 6 & 7 & 7 \\ 0 & 0 & 11 & 12 \\ 0 & 0 & 0 & 16 \end{pmatrix}$$

is upper triangular. Show that the determinant of any $n \times n$ upper triangular matrix A is the product of its entries lying on the diagonal, i.e., $\det A = A_{11}A_{22}\cdots A_{nn}$. **Hint:** Consider a row expansion along the bottom row of A. What do you observe?

Solution: The bottom row of such a matrix is of the form $\begin{pmatrix}0 & 0 & \cdots & 0 & A_{nn}\end{pmatrix}$. Expanding along this row, we find that the determinant is $(-1)^{n+n}A_{nn} = A_{nn}$ times the determinant of the matrix with the last column and last row removed. Now this matrix is an $(n-1)\times(n-1)$ matrix, and is again upper triangular, with diagonal entries $A_{11}, \dots A_{n-1,n-1}$. By expanding along the last row of this matrix, we find that the determinant of this matrix, for exactly the same reasons, is $A_{n-1,n-1}$ times the $(n-2)\times(n-2)$ matrix consisting of the first $(n-2)$ rows and $(n-2)$ columns of A. Continuing in this way, we eventually find that the determinant of A is equal to $A_{nn}A_{n-1,n-1}\cdots A_{22}$ times the determinant of the 1×1 matrix (A_{11}), which itself has determinant A_{11}. The claim follows.

(4) According to our definition in class, B is an inverse for A if $AB = BA = I_n$. Suppose we instead require only that $AB = I_n$. In general algebraic contexts, this will not be enough to guarantee that B is an inverse for A. However, there is enough extra structure in the theory of matrices to conclude in this situation that B is an inverse for A. This problem will guide you through the proof of this fact.

(a) Show that if A, B are square matrices of size $n \times n$ with $AB = I_n$, then A is invertible. (Hint: Use determinants).

(b) Using the notation and results of part (a), show that in fact $B = A^{-1}$ (Hint: Consider the matrix BAB).

Solution:

(a): Taking the determinant of both sides, and using the fact that a determinant of a product is the product of determinants, we have

$$\det(AB) = \det(A)\det(B) = \det(I_n) = 1.$$

Hence, we have $\det A \neq 0$, and so A is invertible.

(b): As A is invertible, there is a unique matrix A^{-1} for which $AA^{-1} = A^{-1}A = I_n$. We claim that in fact $B = A^{-1}$. The only thing we know about B is that

$$AB = I_n.$$

Multiplying both sides of this equation on the left by A^{-1} shows that $B = A^{-1}$, as desired.