INDEFINITE THETA FUNCTIONS OF MORE GENERAL TYPE: NEW RESULTS AND APPLICATIONS

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1. Higher-type indefinite theta functions: Motivation and applications

In the last two lectures, we described several types of theta functions. In particular, we discussed how to form convergent, modular functions by summing terms of the form $p(n)q^{Q(n)}$ (where p is a function possibly depending on $v = \mathrm{Im}(\tau)$) as n ranges over an r-dimensional lattice. We saw that the determination of which functions p are allowed is dictated by a second-order differential equation of Vignéras. The main difficulty is to find choices of p which are both interesting (or include as pieces predetermined examples) and yield convergent sums. If the quadratic form Q is positive definite, then we have great freedom in picking p, as the terms $q^{Q(n)}$ decay rapidly in all directions. As soon as there is a negative value of $Q(n)$, however, there will be exponential growth in some of the terms $q^{Q(n)}$, and so we need to introduce decay or cancellation of terms using the functions p. Zwegers gave very important examples of $(r-1,1)$ forms which are built out of differences of particularly natural solutions to Vignéras' equation. This realization has led to many applications throughout number theory, combinatorics, knot theory, representation theory, and physics.

It is very natural to ask, therefore, whether an analogous theory exists for indefinite theta functions associated to quadratic forms of arbitrary type. In the search of such a theory, it is a good idea to look for motivating examples from related fields, such as physics and representation theory. These alternative perspectives are a guiding force directing one towards objects which are both natural and are built with applications in mind. Zwegers' original motivations in fact did come from such areas, and in particular the first hints of such a theory were given by Göttsche and Zagier in their study of Donaldson invariants for certain 4-manifolds, as well as by the work of Andrews, Watson, and many others on Ramanujan's mock theta functions.

A theory generalizing the indefinite theta functions of Zwegers has begun to blossom in just the past year. This is very exciting, as we are now in a similar situation as we were in directly after Zwegers' thesis was written; namely, there are many new examples of modular-type objects to study which are natural on the number theory side and which have similarly abuntant appearances in other areas of mathematics. Here, we would like to describe a few instances of where these higher-type functions appear, and give some indication as to what shapes they take.

Some early candidates for natural generalizations of Zwegers' functions were certain generalized Appell-Lerch sums which were noticed to be connected to partition functions coming from $\mathcal{N}=4$ supersymmetric gauge theory. Such partition functions also encode Poincaré polynomials of moduli spaces of semi-stable sheaves on the projective plane $\mathbb{P}^2$ with arbitrary rank r, and Manschot showed how these algebro-geometric objects are connected to general Appell-Lerch sums. Another instance where general Appell-Lerch sums were highlighted was in work of Kac and Wakimoto. Extending their work on $s\ell(m|n)^\wedge$ highest weight modules (mentioned in the first lecture in relation to mock theta functions), Kac and Wakimoto also connected functions of the shape in (1) to representations of affine superalgebras.

Definition. Let Q be a positive-dimensional quadratic form for an n_+-dimensional lattice Λ. Now suppose that we choose $m_1, \ldots, m_{n_-}$ in the dual lattice Λ^*. Then for $u = (u_1, \ldots, u_{n_-}) \in \mathbb{C}^{n_-}$, and for $v \in \Lambda^* \otimes \mathbb{C} \cong \mathbb{C}^{n_+}$, we define the **type** (n_+, n_-) **Appell-Lerch sum** by

$$A_{Q,\{m_j\}}(u,v;\tau) := \sum_{k\in\Lambda} \frac{q^{\frac{Q(k)}{2}} e(v\cdot k)}{\prod_{\ell=1}^{n_-}\left(1-q^{k\cdot m_\ell}e(u_\ell)\right)}. \tag{1}$$

Kac and Wakimoto required the vectors m_j to be pairwise orthogonal (which, as we shall hint at below, often makes the functions much simpler), but the applications in gauge theory require more general sets of vectors. As discussed in the first lecture, via geometric series expansions we can essentially think of these functions as indefinite theta series for a quadratic form of type (n_+, n_-). As just two more examples of natural contexts for such indefinite theta functions, we refer the reader to important connections to knot theory made by Hikami and Lovejoy (discussed further in the fourth lecture), as well as series related to Gromov-Witten theory discussed in Section 4.

2. New Realizations

We now describe the recent progress which has been made in understanding the generalized versions of Zwegers' functions. Early modularity results were proven in several cases. For example, Westerholt-Raum used the theory of H-harmonic Maass-Jacobi forms to describe and characterize certain higher-type indefinite theta functions which are modular but of a more complicated type than harmonic Maass forms (these are *higher depth forms*; cf. Section 3). As we shall discuss in Section 4, Bringmann, Zwegers, and the author showed modularity results for special examples with an application to Gromov-Witten theory (and this program set forth by Lau and Zhou was completed later by Bringmann, Kaszian, and the author). More hints were also offered in work by Bringmann, Manschot, and the author, where certain suggestive elliptic transformations of higher-type Appell-Lerch sums were shown, and which gave alternative, direct proofs of certain blow-up formulas.

A few months ago, a major breakthrough was achieved by Alexandrov, Banerjee, Manschot, and Pioline. Their work made the general picture of Zwegers'-type results for indefinite quadratic forms of type $(r-2,2)$ clear, and offered a path to a completely general picture of $(r-s,s)$-type indefinite theta functions. As discussed above, one of the main obscurities in the field before this breakthrough was how to find an appropriate function p which satisfies Vignéras' equation which can play the role that the special function E did in Zwegers' case. In this situation, it turns out that the "right" functions are what they call **double error functions**. These arose from string-theoretic considerations, and are certain Penrose-type integrals. These functions are two-dimensional integrals, but also involve integrating square exponential terms. Explicitly, (although this is a somewhat rewritten form from the original definition, and is stated in their Theorem 3.11), we define for $\alpha \in \mathbb{R}$, $(u_1,u_2) \in \mathbb{R}^2$ with $u_1 \neq 0$, $u_2 \neq \alpha u_1$ the function

$$E_2(\alpha; u_1, u_2) := \int_{\mathbb{R}^2} e^{-\pi(u_1-z_1)^2-\pi(u_2-z_2)^2} \operatorname{sign}(z_2)\operatorname{sign}(z_1+\alpha z_2)dz_1dz_2.$$

Remark. *Constructions such as this can also be considered to be special cases of related results Borcherds used to construct Siegel theta functions.*

Remark. *It is also possible to write the function E from Zwegers' thesis in a similar shape, namely:*

$$E(u) = \int_{\mathbb{R}} e^{-\pi(u-t)^2}\operatorname{sign}(t)dt.$$

This function has the following desirable properties (Exercise 1 will ask you to show the first one):

(1) The function E_2 satisfies Vignéras' equation for $Q(u_1,u_2) = u_1^2 + u_2^2$ with $\lambda = 0$.
(2) Towards infinity, we have

$$E_2(\alpha; u_1, u_2) \sim \operatorname{sgn}(u_2)\operatorname{sgn}(u_1 + \alpha u_2).$$

Now, given a quadratic form Q of type $(r-2,2)$ and vectors c_1, c_2 with $Q(c_1), Q(c_2) < 0$ and $\Delta(c_1,c_2) := Q(c_1)Q(c_2) - B(c_1,c_2)^2 > 0$, we can define the **boosted error function**

$$E_2(c_1,c_2;x) := E_2\left(\frac{B(c_1,c_2)}{\sqrt{\Delta(c_1,c_2)}}; \frac{B(c_{1\perp 2},x)}{\sqrt{Q(c_{1\perp 2})}}, \frac{B(c_2,x)}{\sqrt{Q(c_2)}}\right),$$

where

$$c_{1\perp 2} := c_1 - \frac{B(c_1,c_2)}{Q(c_2)}c_2.$$

The point is that this "boosted" function now satisfies Vignéras' equation with $\lambda = 0$ for our quadratic form Q, and towards infinity it grows like a product of two sign functions:

$$E(c_1,c_2;x) \sim \operatorname{sgn}(B(c_1,x))\operatorname{sgn}(B(c_2,x)). \tag{2}$$

Similar to the case in Zwegers' thesis, convergent indefinite theta series of type $(r-2,2)$ can be build out of pairs of these boosted functions; for example (2) can be used to show that these indefinite theta series have "holomorphic parts" with a shape

$$\sum_{n\in\mathbb{Z}^r} \left(\operatorname{sgn}(B(c_1,n)) - \operatorname{sgn}(B(c_1',n))\right)\left(\operatorname{sgn}(B(c_2,n)) - \operatorname{sgn}(B(c_2',n))\right) q^{Q(n)}.$$

As another example, Alexandrov, Banerjee, Manschot, and Pioline gave modular completions of generalized μ-functions such as ($\zeta_j := e(z_j)$)

$$\mu_{2,1}(z_1,z_2;\tau) := \frac{1}{2} + \frac{\zeta_1^{\frac{1}{2}}}{\vartheta(z_2;\tau)} \sum_{n_1,n_2\in\mathbb{Z}} \frac{\zeta_2^{n_1+2n_2} q^{n_1^2+n_2^2+n_1n_2+2n_1n_2}}{1-\zeta_1 q^{2n_1+n_2}}.$$

Following these results of Alexandrov, Banerjee, Manschot, and Pioline, using a roadmap which they described at the end of their work, general Zwegers'-type indefinite theta functions were elucidated by Nazaroglu for $(n-r,r)$ quadratic forms (although the "cuspidal" case hasn't been worked out yet). Exercises 4 and 5 will ask you to work out some properties of some higher-type error functions which can be used to construct generalized examples.

3. Higher depth mock modular forms

The functions constructed by Alexandrov, Banerjee, Manschot, and Pioline should play an important role in number theory, since they have many applications, as discussed above. It is natural to ask what spaces these modular-type objects live in. The results described above show that these functions transform like modular forms, but like harmonic Maass forms, they should be "completions" of holomorphic functions which are built out of "simpler" functions. It turns out that the correct definition for such spaces is the following.

Definition (Zagier-Zwegers,Westerholt-Raum)**.** Let $M_k^0 := M_k$, the space of modular forms of weight k. For $d>0$, define the space M_k^d of depth d forms to be the space of functions which transform like modular forms and whose images under ξ_k lie in

$$\sum_{\ell} \overline{M_\ell} \otimes M_{k-\ell}^{d-1}.$$

For example, the depth 1 forms are just the mixed mock modular forms, briefly discussed in the last two lectures. The reason that such indefinite theta functions built using Vignéras' theorem often lie in such spaces follows by a useful fact noticed by Vignéras. Namely, the **Maass lowering operator**

$$L_k := -2iv^2\frac{\partial}{\partial\overline{\tau}}$$

is essentially the shadow operator ξ_k and intertwines in a beautiful way with indefinite theta functions. In particular, if p satisfies Vigneras' equation with eigenvalue λ, then

$\mathcal{E}-\lambda$ sends p to an eigenfunction with eignvalue $\lambda-2$, and the theta function built out of this function is (up to a fixed constant) the lowering operator applied on the indefinite theta function for p. Exercise 2 will ask you to compute the action of this operator on the double error function E_2, which directly shows how the type $(r-2,2)$ indefinite theta functions above are depth 2 forms.

4. An example from Gromov-Witten theory

Several examples which arises from Gromov-Witten theory of elliptic orbifolds were studied by Bringmann-R.-Zwegers and Bringmann-Kaszian-R., and are motivated by their connections to mirror symmetry and applications of modularity results in extending certain potentials to global moduli spaces. Most of the functions which arise can be written very neatly in terms of the simple indefinite theta function

$$F(z_1,z_2,z_3;\tau):=q^{-\frac{1}{8}}\zeta_1^{-\frac{1}{2}}\zeta_2^{\frac{1}{2}}\zeta_3^{\frac{1}{2}}\left(\sum_{k>0,\,\ell,m\geq 0}+\sum_{k\leq 0,\,\ell,m<0}\right)(-1)^k q^{\frac{1}{2}k^2+\frac{1}{2}k+k\ell+km+\ell m}\zeta_1^k\zeta_2^\ell\zeta_3^m.$$

The key result which makes the applications to Gromov-Witten theory possible is then the following curious identity ($|\mathrm{Im}(z_2)|,|\mathrm{Im}(z_3)|<\mathrm{Im}(\tau)$):

(3)

$$F(z_1,z_2,z_3;\tau)=i\vartheta(z_1;\tau)\mu(z_1,z_2;\tau)\mu(z_1,z_3;\tau)-\frac{\eta^3(\tau)\vartheta(z_2+z_3;\tau)}{\vartheta(z_2;\tau)\vartheta(z_3;\tau)}\mu(z_1,z_2+z_3;\tau).$$

Although this identity can be proven using calculations of the corresponding elliptic transformations together with Liouville's theorem (Exercise 3 will ask you to explore some of the steps of these calculations), such "elementary" considerations prove most of the main results in Chapter 1 of Zwegers' thesis as well. As was the case for the μ-function, the most interesting task it to *find* such a decomposition in the first place.

Note that this example is also of a much simpler form than the general higher-type indefinite theta functions defined above; indeed, it splits into products of modular and mock modular forms. The general philosophy about such situations is that these special scenarios typically happen when the quadratic form is degenerate (as in this case), the vectors such as those in (1) have orthogonality relations, and the quadratic form has inordinately many symmetries in its variables.

5. Exercises

(1) Show that the double error function $E_2(\alpha;u_1,u_2)$ satisfies Vignéras' equation for $Q(u_1,u_2)=u_1^2+u_2^2$ with $\lambda=0$. That is, show

$$\left(\partial_{w_1}^2+\partial_{w_2}^2+2\pi\left(w_1\partial_{w_1}+w_2\partial_{w_2}\right)\right)E_2\left(\alpha;w_1,w_2\right)=0,$$

(2) Show that

$$\partial_{w_2} E_2(\alpha; w_1, w_2) = \frac{2}{\sqrt{1+\alpha^2}} e^{-\frac{\pi(w_2+\alpha w_1)^2}{1+\alpha^2}} E\left(\frac{\alpha w_2 - w_1}{\sqrt{1+\alpha^2}}\right), \text{ and}$$
$$\partial_{w_1} E_2(\alpha; w_1, w_2) = 2e^{-\pi w_1^2} E(w_2) + \frac{2\alpha}{\sqrt{1+\alpha^2}} e^{-\frac{\pi(w_2+\alpha w_1)^2}{1+\alpha^2}} E\left(\frac{\alpha w_2 - w_1}{\sqrt{1+\alpha^2}}\right).$$

Using the discussion in Section 3, deduce what the shadows (or lowering operators of) indefinite theta functions built out of the boosted functions E_2 are.

(3) In this problem, you will consider some of the first steps towards proving (3). The idea is to consider both sides of the equation as functions of z_3, and compute their elliptic transformations and locations of poles and residues. Showing that these all match will then show, by a standard application of Liouville's theorem (as it is commonly used in elliptic function theory) that the identity holds.

Here you will prove the desired transformation for the left-hand-side. First, show, using geometric series, that for $|y_3| < v$ ($z_3 = x_3 + iy_3$), we can write the left-hand-side $f_L(z_3)$ as

$$q^{-\frac{1}{8}} \zeta_1^{-\frac{1}{2}} \zeta_2^{\frac{1}{2}} \zeta_3^{\frac{1}{2}} \left(\sum_{\substack{k,\ell\in\mathbb{Z} \\ k>0,\ \ell\geq 0}} - \sum_{\substack{k,\ell\in\mathbb{Z} \\ k\leq 0,\ \ell<0}} \right) \frac{(-1)^k q^{\frac{1}{2}k^2+\frac{1}{2}k+k\ell} \zeta_1^k \zeta_2^\ell}{1-\zeta_3 q^{k+\ell}}$$
$$= q^{-\frac{1}{8}} \zeta_1^{-\frac{1}{2}} \zeta_2^{\frac{1}{2}} \zeta_3^{\frac{1}{2}} \sum_{k,\ell\in\mathbb{Z}} \rho(k-1,\ell) \frac{(-1)^k q^{\frac{1}{2}k^2+\frac{1}{2}k+k\ell} \zeta_1^k \zeta_2^\ell}{1-\zeta_3 q^{k+\ell}},$$

where

$$\rho(k,\ell) := \begin{cases} 1 & \text{if } k,\ell \geq 0, \\ -1 & \text{if } k,\ell < 0, \\ 0 & \text{otherwise.} \end{cases}$$

For $0 < y_2 < v$, show that

$$\zeta_2^{-\frac{1}{2}} \vartheta(z_1)\mu(z_1,z_2) = \sum_{k,\ell\in\mathbb{Z}} \rho(k,\ell) \frac{(-1)^k q^{\frac{1}{2}k^2+\frac{1}{2}k+k\ell} \zeta_1^k \zeta_2^\ell}{1-\zeta_3 q^{k+\ell+1}} - \zeta_3 \sum_{k,\ell\in\mathbb{Z}} \rho(k,\ell) \frac{(-1)^k q^{\frac{1}{2}k^2+\frac{3}{2}k+1+k\ell+\ell} \zeta_1^k \zeta_2^\ell}{1-\zeta_3 q^{k+\ell+1}}.$$

Using the easily checked identity

$$\rho(k,\ell) = \rho(k-1,\ell) + \delta_k,$$

($\delta_k = 1$ if $k=0$ and $\delta_k = 0$ otherwise) in the first sum and replacing k by $k-1$ in the second, compute that

$$\zeta_2^{-\frac{1}{2}} \vartheta(z_1)\mu(z_1,z_2) = q^{-\frac{3}{8}} \zeta_1^{\frac{1}{2}} \zeta_2^{-\frac{1}{2}} \zeta_3^{-\frac{1}{2}} f_L(z_3+\tau) - i\zeta_2^{-1} \frac{\eta^3 \vartheta(z_2+z_3)}{\vartheta(z_2)\vartheta(z_3)} + q^{\frac{1}{8}} \zeta_1^{-\frac{1}{2}} \zeta_2^{-\frac{1}{2}} \zeta_3^{\frac{1}{2}} f_L(z_3).$$

(Hint: use the identity

$$\sum_{\ell\in\mathbb{Z}} \frac{\zeta_2^\ell}{1-\zeta_3 q^{\ell+1}} = -i\zeta_2^{-1}\frac{\eta^3\vartheta(z_2+z_3)}{\vartheta(z_2)\vartheta(z_3)}.)$$

Conclude that

$$f_L(z_3)+q^{-\frac{1}{2}}\zeta_1\zeta_3^{-1}f_L(z_3+\tau) = q^{-\frac{1}{8}}\zeta_1^{\frac{1}{2}}\zeta_3^{-\frac{1}{2}}\vartheta(z_1)\mu(z_1,z_2)+iq^{-\frac{1}{8}}\zeta_1^{\frac{1}{2}}\zeta_2^{-\frac{1}{2}}\zeta_3^{-\frac{1}{2}}\frac{\eta^3\vartheta(z_2+z_3)}{\vartheta(z_2)\vartheta(z_3)}.$$

(4) Define a generalized error function ($N\geq 3$) $E_N\colon \mathbb{R}^{\frac{N(N-1)}{2}}\times\mathbb{R}^N\to\mathbb{R}$ by

$$E_N(\alpha;w) := \int_{\mathbb{R}^N} \mathrm{sign}(t_1)\mathrm{sign}(t_2+\alpha_1 t_1)\mathrm{sign}(t_3+\alpha_2 t_1+\alpha_3 t_2)\cdots\mathrm{sign}\left(t_N+\ldots+\alpha_{\frac{N(N-1)}{2}}t_{N-1}\right)e^{-\pi\|t-w\|_2^2}dt,$$

where $\|a\|_2 := \sqrt{a^Ta}$ denotes the Euclidian norm. Show that, as $\lambda\to\infty$,

$$E_3(\alpha;\lambda w)\sim \mathrm{sign}\left(w_1\right)\mathrm{sign}\left(w_2+\alpha_1 w_1\right)\mathrm{sign}\left(w_3+\alpha_2 w_1+\alpha_3 w_2\right).$$

Hint: Change variables to show

$$E_3\left(\alpha;w_1,w_2,w_3\right) = \int_{\mathbb{R}^3} e^{-\pi t^TMt}\mathrm{sign}\left(t_1+w_1\right)\mathrm{sign}\left(t_2+v_2\right)\mathrm{sign}\left(t_3+v_3\right)dt,$$

where $t:=(t_1,t_2,t_3)^T, v_2:=w_2+\alpha_1 w_1, v_3:=w_3+\alpha_2 w_1+\alpha_3 w_2$, and

$$M := \begin{pmatrix} 1+(\alpha_1\alpha_3-\alpha_2)^2+\alpha_1^2 & -\alpha_1-\alpha_3(\alpha_1\alpha_3-\alpha_2) & \alpha_1\alpha_3-\alpha_2 \\ -\alpha_1-\alpha_3(\alpha_1\alpha_3-\alpha_2) & \alpha_3^2+1 & -\alpha_3 \\ \alpha_1\alpha_3-\alpha_2 & -\alpha_3 & 1 \end{pmatrix}.$$

Now consider the difference

$$\begin{aligned}&E_3(\alpha;\lambda w_1,\lambda w_2,\lambda w_3)-\mathrm{sign}\left(w_1\right)\mathrm{sign}\left(w_2+\alpha_1 w_1\right)\mathrm{sign}\left(w_3+\alpha_3 w_2+\alpha_2 w_1\right)\\ &= \int_{\mathbb{R}^3} e^{-\pi t^TMt}\big(\mathrm{sign}\left(t_1+\lambda v_1\right)\mathrm{sign}\left(t_2+\lambda v_2\right)\mathrm{sign}\left(t_3+\lambda v_3\right)-\mathrm{sign}\left(v_1\right)\mathrm{sign}\left(v_2\right)\mathrm{sign}\left(v_3\right)\big)dt.\end{aligned}$$

(5) Prove the analogue of the Vignéras equation in Exercise 1 for E_3. Specifically, show that

$$\sum_{j=1}^{3}\left(\partial_{w_j}^2+2w_j\partial_{w_j}\right)E_3(\alpha;w)=0.$$