Trinity College Dublin
Coláiste na Tríonóide, Baile Átha Cliath
The University of Dublin

Faculty of Engineering, Mathematics and Science

School of Mathematics

GROUPS **Trinity Term 2016**

MA1132: Advanced Calculus SAMPLE EXAM, Solutions

DAY PLACE TIME

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Instructions to Candidates:

Attempt all questions. All questions will be weighted equally.

Materials Permitted for this Examination:

Formulae and Tables are available from the invigilators, if required.

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used. This is a closed-book exam, so no notes or other study materials are allowed.

You may not start this examination until you are instructed to do so by the Invigilator.

1. Suppose a curve is given by the parametric equations

$$\begin{cases} x = \frac{t^2}{2} \\ y = \frac{4t^{\frac{5}{2}}}{5} \\ z = \frac{2t^3}{3}, \end{cases}$$

where $t > 0$.

(a) Find the unit tangent vector $T(t)$ to the curve as a vector-valued function of t.

(b) Compute the unit normal vector $N(t)$.

(c) Compute the unit binormal vector $B(t)$.

(d) Compute the curvature as a function of t.

Solution:

(a): This curve is the graph of the vector-valued function

$$r(t) = \left(\frac{t^2}{2}, \frac{4t^{\frac{5}{2}}}{5}, \frac{2t^3}{3} \right),$$

which has derivative

$$r'(t) = (t, 2t^{\frac{3}{2}} 2t^2).$$

The norm of this derivative is

$$|r'(t)| = \sqrt{t^2 + 4t^3 + 4t^4} = t\sqrt{1 + 4t + 4t^2} = t(1 + 2t).$$

Here, we used the fact that $t > 0$ to avoid writing absolute values when we took the square root. Thus,

$$T(t) = \frac{r'(t)}{|r'(t)|} = \frac{1}{(1 + 2t)}(1, 2\sqrt{t}, 2t).$$

(b): We take the derivative of our answer in (a):

$$T'(t) = \frac{1}{(1 + 2t)^2} \left(-2, \frac{1 - 2t}{\sqrt{t}}, 2 \right),$$

and compute the norm of this vector,

$$|T'(t)| = \sqrt{\frac{4 + \frac{(1-2t)^2}{t} + 4}{(1 + 2t)^4}} = \frac{1}{(1 + 2t)^2\sqrt{t}}\sqrt{8t + 1 - 4t + 4t^2} = \frac{1}{(1 + 2t)\sqrt{t}}.$$

The quotient is

$$N(t) = \frac{T'(t)}{|T'(t)|} = \frac{1}{1+2t}(-2\sqrt{t}, 1-2t, 2\sqrt{t}).$$

(c): The unit binormal vector is the cross product of the vectors from (a) and (b):

$$\begin{aligned} B(t) &= T(t) \times N(t) \\ &= \frac{1}{(1+2t)^2}(1, 2\sqrt{t}, 2t) \times (-2\sqrt{t}, 1-2t, 2\sqrt{t}) \\ &= \frac{1}{(1+2t)^2} \det \begin{pmatrix} i & j & k \\ 1 & 2\sqrt{t} & 2t \\ -2\sqrt{2} & 1-2t & 2\sqrt{t} \end{pmatrix} = \frac{1}{(1+2t)}(2t, -2\sqrt{t}, 1). \end{aligned}$$

(d): There are many formulas for the curvature, and you can use whichever you prefer. Since we've already computed T' and r', it is convenient to use the formula

$$\kappa(t) = \frac{|T'(t)|}{|r'(t)|} = \frac{1}{(1+2t)^2 t^{\frac{3}{2}}}.$$

2. (a) Find the directional derivative of $f(x, y) = \cos(xy) - x$ at the point $(0, 1)$ in the direction of the vector $(1, 3)$.

(b) Consider the function

$$f(x, y, z) = \sqrt{\frac{z - xy}{x + z}}$$

and the point $P = (1, 2, 3)$. At the point P, in which direction does f increase the fastest? That is, find a vector pointing in the direction of largest increase. Also find the magnitude of this rate of increase in that direction.

(c) Use the chain rule to show that the gradient of a $\mathcal{C}^1$ (continuous first order partial derivatives) function $f(x, y)$ at any point $P \in \mathbb{R}^2$ is perpendicular to the level curve of f through P, if it isn't equal to 0.

Solution:

(a): We must first normalize the vector $(1, 3)$ to make it a unit vector. Its length is $|(1, 3)| = \sqrt{1+9} = \sqrt{10}$, and the unit vector pointing in the same direction is $u = \frac{1}{\sqrt{10}}(1, 3)$. We also compute the gradient of f:

$$\nabla f(x, y) = (f_x, f_y) = (-y \sin(xy) - 1, -x \sin(xy)),$$

which at $P = (0, 1)$ is

$$\nabla(P) = (-1, 0).$$

Thus, the directional derivative of f in the direction of $(1, 3)$ is

$$D_u f(P) = \nabla(P) \cdot u = \frac{1}{\sqrt{10}}(-1, 0) \cdot (1, 3) = -\frac{1}{\sqrt{10}}.$$

(b): Since this number will show up in all of the derivatives, for convenience we first record the evaluation

$$f(P) = \left(\sqrt{\frac{z - xy}{x + z}}\right)\Bigg|_P = \frac{1}{2}.$$

We then compute the partial derivatives:

$$f_x = \frac{1}{2\sqrt{\frac{z-xy}{x+z}}} \frac{-y(x + z) - (z - xy)}{(x + z)^2},$$

which at P is

$$f_x(P) = \frac{-2 \cdot 4 - 1}{4^2} = -\frac{9}{16}.$$

Similarly,

$$f_y = \frac{1}{2\sqrt{\frac{z-xy}{x+z}}} \frac{-x(x + z)}{(x + z)^2},$$

which at P is

$$f_y(P) = -\frac{4}{16} = -\frac{1}{4},$$

and

$$f_z = \frac{1}{2\sqrt{\frac{z-xy}{x+z}}} \frac{(x + z) - (z - xy)}{(x + z)^2},$$

which evaluates to

$$f_z(P) = \frac{3}{16}$$

Thus, the gradient at P is

$$\nabla f(P) = \frac{1}{16}(-9, -4, 3),$$

which has norm $\frac{1}{16}\sqrt{81 + 16 + 9} = \frac{\sqrt{106}}{16}$. Thus, f increases the fastest in the direction of $\frac{1}{16}(-9, -4, 3)$, with a rate of change equal to $\frac{\sqrt{106}}{16}$.

(c): Suppose the value of f at a point P is c; then the level curve passing through P is the curve described by $f(x,y) = c$. We then take a smooth parameterization $x = x(s)$, $y = y(s)$ of this curve, which is parameterized by the arc length s. Then, differentiating the equation

$$f(x,y) = 0$$

with respect to s using the chain rule gives

$$\frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} = 0.$$

But this is just the expansion of the dot product

$$\nabla f \cdot \left(\frac{dx}{ds}, \frac{dy}{ds}\right) = \nabla f \cdot T,$$

where T is the unit tangent vector to the level curve. Thus, if $\nabla f \neq 0$, then the gradient is orthogonal to the tangent line through the curve at that point, which is what is meant by saying that the gradient is orthogonal to the curve at that point.

3. (a) Consider the function $f(x,y) = x^2 - x + \cos(xy)$. Find all critical points of f, and decide which are local maxima, local minima, saddle points, or for which the second derivative test is inconclusive.

(b) The Extreme Value Theorem guarantees that the function $f(x,y) = x^3 + y^3$ has a global maximum value and a global minimum value on the circle $x^2 + y^2 = 1$. Use the method of Lagrange multipliers to find these values.

Solution:

(a): We take the first and second order partial derivatives:

$$f_x = 2x - 1 - y\sin(xy), \qquad f_y = -x\sin(xy),$$

$$f_{xx} = 2 - y^2\cos(xy), \qquad f_{yy} = -x^2\cos(xy), \qquad f_{xy} = f_{yx} = -\sin(xy) - xy\cos(xy).$$

These all exist everywhere, so the critical points will occur where $f_x = f_y = 0$. Now if $f_y = 0$, then we have either $x = 0$ or $\sin xy = 0$. But if $x = 0$, then the equation $f_x = 0$ gives $-1 - 0 = 0$, which is a contradiction. Thus, $x \neq 0$, but $\sin(xy) = 0$. Plugging this into $f_x = 0$ gives $2x - 1 = 0$, and so $x = \frac{1}{2}$. Finally, the equation $\sin(xy) = 0$ becomes $\sin\left(\frac{y}{2}\right) = 0$, which implies that $\frac{y}{2}$ is an integer multiple of π, or that $y = 2n\pi$ for some integer n. Thus, there is a critical point at each point

$$\left(\frac{1}{2}, 2n\pi\right),$$

where n ranges over all integers. To classify these points, there are two cases to consider. In either case, $\sin(xy) = 0$ by choice. Now, if n is even, then $\cos(xy) = \cos(n\pi) = 1$, and if n is odd, then $\cos(xy) = \cos(n\pi) = -1$. Thus, in the first case, when n is even, we have $f_{xx} = 2 - 4n^2\pi^2$, $f_{yy} = -\frac{1}{4}$, $f_{xy} = -n\pi$, and so

$$D = f_{xx}f_{yy} - f_{xy}^2 = -\frac{1}{2} + n^2\pi^2 - n^2\pi^2 = -\frac{1}{2} < 0,$$

and so there is a saddle point in each of these places. For odd n, $f_{xx} = 2 + 4n^2\pi^2$, $f_{yy} = \frac{1}{4}$, $f_{xy} = n\pi$, and so

$$D = f_{xx}f_{yy} - f_{xy}^2 = \frac{1}{2} + n^2\pi^2 - n^2\pi^2 = \frac{1}{2} > 0,$$

and $f_{xx} = 2 + 4n^2\pi^2 > 0$, so there is a local minimum. Thus, on the vertical line $x = \frac{1}{2}$, the function f has an alternating sequence of saddle points and local minima separated by a common distance of 2π, and these are all of the critical points of f.

(b): We compute that

$$\nabla f = (f_x, f_y) = 3(x^2, y^2).$$

This only vanishes when $x = y = 0$, which doesn't happen at the circle. Thus all local extreme of f on the circle $g(x, y) = x^2 + y^2 - 1 = 0$ where $\nabla f = \lambda \nabla g$ for some number λ, and as the circle has no boundary the global extrema must occur at these points. Now $\nabla g = 2(x, y)$, and so by factoring out the 2 and the 3, we have to solve the system of equations

$$\begin{cases} x^2 = \lambda x \\ y^2 = \lambda y \\ x^2 + y^2 = 1. \end{cases}$$

Firstly, if $x = 0$ or $y = 0$, the first two equations are solvable for some λ, giving us the points $(\pm 1, 0), (0 \pm 1)$ to check. If neither x nor y is 0, then solving the first two equations gives $x = \lambda = y$, and so we want the points on the intersection of the circle with the line $y = x$, which are the points $\pm\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$. The values at these points are ± 1 and $2 \cdot 2^{-\frac{3}{2}} = \frac{1}{\sqrt{2}}$. Thus, the maximum value of f on the circle is 1, and the minimum value is -1.

4. (a) Consider the cone given by the equation $z = \sqrt{x^2 + y^2}$ bounded above by the plane $z = 5$. Find parametric equations for this surface.

(b) Write the general equation for the surface area of a surface $x = x(u, v), y = y(u, v), z = z(u, v)$ determined by $(u, v) \in R$, where R is a region in the u-v plane.

(c) Use (a) and (b) to compute the surface area of this cone.

Solution: (a) We can use polar coordinates to parameterize this surface (but we will rename them as u, v). That is, we can set

$$\begin{cases} x = u\cos v \\ y = u\sin v \\ z = u, \end{cases}$$

and the cone is the image of the rectangle $[0, 5] \times [0, 2\pi)$ in the u-v plane.

(b) The surface area will be given by

$$\iint_R \left| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right| dA,$$

where $r(u, v) = (x(u, v), y(u, v), z(u, v))$.

(c) In our special case, we have

$$r(u, v) = (u\cos v, u\sin v, u),$$

$$r_u = (\cos v, \sin v, 1),$$

and

$$r_v = (-u\sin v, u\cos v, 0).$$

Then the cross product is

$$r_u \times r_v = \det \begin{pmatrix} i & j & k \\ \cos v & \sin v & 1 \\ -u\sin v & u\cos v & 0 \end{pmatrix}$$
$$= (-u\cos v, -u\sin v, u(\cos^2 v + \sin^2 v)) = u(-\cos v, -\sin v, 1).$$

This has norm

$$|r_u \times r_v| = u\sqrt{\cos^2 v + \sin^2 v + 1} = u\sqrt{2},$$

where we used the fact that u is positive in the region we are interested in. Thus, the surface area is the integral of this quantity over R, that is,

$$\sqrt{2}\int_0^{2\pi}\int_0^5 ududv = 2\pi\sqrt{2}\int_0^5 udu = 25\pi\sqrt{2}.$$