TUTORIAL 7 SOLUTIONS

MA1132: ADVANCED CALCULUS, HILARY 2017

(1) Using the method of Lagrange multipliers, find the point on the plane $x-y+3z=1$ closest to the origin.

Solution: The distance of an arbitrary point (x, y, z) from the origin is $d = \sqrt{x^2+y^2+z^2}$. It is geometrically clear that there is an absolute minimum of this function for (x, y, z) lying on the plane. To find it, we instead minimize the function

$$d^2 = f(x, y, z) = x^2 + y^2 + z^2$$

subject to the constraint $g(x, y, z) = 0$ where $g(x, y, z) = x - y + 3z - 1$. The gradients of these two functions are $\nabla f = (2x, 2y, 2z)$, $\nabla g = (1, -1, 3)$. Since $\nabla g \neq 0$ ever, the absolute minimum of the distance function we are looking for will occur at a point where

$$\nabla f = \lambda g, \qquad g = 0.$$

Getting rid of the 2's in ∇f (which all get absorbed into the dummy constant λ) and setting components of the gradient equation equation, we obtain the system of equations

$$\begin{cases} x = \lambda \\ y = -\lambda \\ z = 3\lambda \\ x - y + 3z = 1. \end{cases}$$

Solving the first three equations gives $y = -x$, $z = 3x$. Plugging these into the equation of the plane gives $x+x+9x = 11x = 1$, and so the point we are looking for is $x = 1/11$, $y = -1/11$, $z = 3/11$.

(2) Compute the double integral

$$\int_0^1 \int_0^{\sqrt{\log 2}} xye^{x^2}\,dxdy.$$

Solution: We evaluate:

$$\int_0^1 \int_0^{\sqrt{\log 2}} xye^{x^2} dxdy = \int_0^1 y \left(\int_0^{\sqrt{\log 2}} xe^{x^2} dx \right) dy$$
$$= \int_0^1 y \left[\frac{e^{x^2}}{2} \right]_{x=0}^{\sqrt{\log 2}} dy$$
$$= \int_0^1 y \left(1 - \frac{1}{2} \right) dy = \int_0^1 \frac{y}{2} dy$$
$$= \left[\frac{y^2}{4} \right]_0^1 = \frac{1}{4}.$$

(3) Find the volume under the surface $z = \frac{x}{y}$ and above the rectangular region $R = [0,2] \times [1,3]$ in the x-y plane.

Solution: The function $f(x,y) = x/y$ is always non-negative on $[0,2] \times [1,3]$, and so the double integral $\iint_R f(x,y)dA$ which in general gives the net volume, gives the actual volume in this case. Thus, the volume we want to compute is

$$V = \int_1^3 \int_0^2 \frac{x}{y} dxdy$$
$$= \int_1^3 y^{-1} \left(\int_0^2 x \right) dxdy$$
$$= \int_1^3 y^{-1} \left[\frac{x^2}{2} \right]_0^2 dy$$
$$= \int_1^3 \frac{2}{y} dy = [2 \log y]_1^3 = 2 \log 3.$$

Advanced Problem: Suppose that $A = (A_{ij})$ is a symmetric, real-valued $n \times n$ matrix. Define a function $f : \mathbb{R}^n \to \mathbb{R}$ by the dot product $f(x) = x \cdot Ax$. Show that the largest and smallest values of f on the unit sphere $\{x \in \mathbb{R}^n | |x| = 1\}$ are the largest and smallest (real) eigenvalues of A. (Hints: Use the method of Lagrange multipliers. What is ∇f? What is ∇g where $g(x) = x \cdot x - 1$? Try writing down a few examples for small n first.) Deduce that every real-valued, symmetric $n \times n$ matrix has at least one real eigenvalue.

Solution: The function f is clearly continuous, and the sphere is a compact set, so by the Extreme Value Theorem, f has a global max and min on the sphere. We want to extremize f subject to the constraint $g = 0$ with $g(x) = x \cdot x - 1$. Writing this out using coordinates, say $x = (x_1, \dots, x_n)$, $g(x_1, \dots, x_n) = x_1^2 + \dots + x_n^2 - 1$, and so $\nabla g(x) = (2x_1, \dots, 2x_n) = 2x$. On the sphere, ∇g thus doesn't vanish, and so our extrema must occur at points where $\nabla f = \lambda g$ and $g = 0$ (there are no boundary points of the sphere to consider).

Now we compute the gradient ∇f. First, we write out f explicitly using the coordinates of x and matrix entries A_{ij}. Specifically, we see that $f(x) = \sum_{i=1}^n x_i(Ax)_i = \sum_{i=1}^n x_i \sum_{j=1}^n A_{ij}x_j = \sum_{i=1}^n \sum_{j=1}^n A_{ij}x_i x_j$. Now suppose we take the derivative of the last expression with respect to x_i. There are two types of pieces which will survive the differentiation. There is the piece $A_{ii}x_i^2$, which has derivative $2A_{ii}x_i$. For each $j \neq i$, there are also two pieces $A_{ij}x_i x_j$ and $A_{ji}x_j x_i$. Since the matrix A is symmetric, $A_{ij} = A_{ji}$, and the sum of these two pieces is $2A_{ij}x_i x_j$, which has derivative $2A_{ij}x_j$. Thus,

$$\frac{\partial f}{\partial x_i} = 2A_{ii}x_i + \sum_{\substack{j=1 \\ i\neq j}}^n 2A_{ij}x_j = 2\sum_{j=1}^n A_{ij}x_j = 2(Ax)_i.$$

Putting all the derivatives together implies that $\nabla f = 2Ax$. Thus, the equation $\nabla f = \lambda \nabla g$ becomes $Ax = \lambda x$, which implies that x is an eigenvector of A with eigenvalue λ. Hence, the maximum and minimum values of f on the sphere occur at eigenvectors of A. At such an eigenvector, $f(x) = x \cdot Ax = x \cdot (\lambda x) = \lambda x \cdot x = \lambda$, as the vector x lies on the sphere. Thus, the maximum and minimum values of f on the sphere are the largest and smallest eigenvlaues of A. In particular, since the Extreme Value Theorem guarantees that such a max and a min exist, that A has at least one real eigenvalue. This is the main step in proving a very important result, known as the *real spectral theorem*. That is, you can inductively use this result to show that all eigenvalues of A are real. This is somewhat like to prove the Fundamental Theorem of Algebra (that all complex-coefficient polynomials of degree $n \geq 1$ have n complex roots) follows easily after one shows the much more difficult step that every such polynomial has at least one complex root.