# TUTORIAL 5 SOLUTIONS

MA1132: ADVANCED CALCULUS, HILARY 2017

(1) Use the chain rule to find $\frac{dz}{dt}$ when

$$z = \sin(xy) + e^{xy}, \qquad x = t^2, \qquad y = t.$$

Check your answer by directly plugging in $x = t^2$ and $y = t$ into $z$ and taking the derivative with respect to $t$.

**Solution:** The chain rule in this situation states that

$$\begin{aligned} \frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\ &= (y\cos(xy) + ye^{xy}) \cdot (2t) + (x\cos(xy) + xe^{xy}) \cdot 1 \\ &= (t\cos(t^3) + te^{t^3}) \cdot (2t) + t^2\cos(t^3) + t^2e^{t^3} = 3t^2\left(\cos(t^3) + e^{t^3}\right). \end{aligned}$$

To find $\frac{dz}{dt}$ directly, we can also plug in $x = t^2$, $y = t$ into the definition of $z$ to find that

$$z = \sin(t^3) + e^{t^3},$$

so that

$$\frac{dz}{dt} = 3t^2\left(\cos(t^3) + e^{t^3}\right),$$

which agrees with the computation above.

(2) Suppose that

$$w = \frac{xy}{x^2 + z^2}, \quad x = r + s, \quad y = r - s, \quad z = 1.$$

Use the chain rule to find $\frac{\partial w}{\partial s}$.

**Solution:** In this case, the chain rule states that

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}.$$

Since $\frac{\partial x}{\partial s} = 1$, $\frac{\partial y}{\partial s} = -1$, and $\frac{\partial z}{\partial s} = 0$, this becomes

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} - \frac{\partial w}{\partial y}.$$

These two derivatives are computed via the quotient rule to be:

$$\frac{\partial w}{\partial x} = \frac{y(x^2 + z^2) - 2x^2y}{(x^2 + z^2)^2} = \frac{z^2y - x^2y}{(x^2 + z^2)^2} = \frac{(r - s) - (r + s)^2(r - s)}{((r + s)^2 + 1)^2},$$

$$\frac{\partial w}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x}{x^2+z^2}\cdot y\right) = \frac{x}{(x^2+z^2)} = \frac{r+s}{(r+s)^2+1}.$$

Hence,

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} - \frac{\partial w}{\partial y} = \frac{(r-s)-(r+s)^2(r-s)}{((r+s)^2+1)^2} - \frac{r+s}{(r+s)^2+1}.$$

This answer is already acceptable (and useful if a point to plug in and evaluate were given), but it can also be written slightly more nicely as

$$\frac{\partial w}{\partial s} = -2\cdot\frac{r^3+2r^2s+rs^2+s}{r^2+2rs+s^2+1)^2}.$$

(3) Suppose that $f(x,y) = x\cos y - y\sin x$ and $(x_0, y_0) = (\pi/2, \pi)$. Find the directional derivatives of $f$ at $(x_0, y_0)$ in the directions of the following two vectors:
(a) $(3/5, -4/5)$,
(b) $(1, 2)$.

**Solution:**

We first compute the gradient in general:

$$\nabla f = (f_x, f_y) = (\cos y - y\cos x, -x\sin y - \sin x),$$

and at the specified point:

$$\nabla f\left(\frac{\pi}{2}, \pi\right) = (-1, -1).$$

Now we can find the directional derivatives we are looking for:

a). This is already a unit vector, as $|(3/5, -4/5)| = 1$ (as $3^2+4^2=5^2$). Thus, we can take $u = (3/5, -4/5)$ and the directional derivative we want to compute is

$$D_u\left(\frac{\pi}{2}, \pi\right) = \nabla f\left(\frac{\pi}{2}, \pi\right)\cdot u = (-1,-1)\cdot\left(\frac{3}{5}, -\frac{4}{5}\right) = -\frac{3}{5}+\frac{4}{5} = \frac{1}{5}.$$

b). This is not, a unit vector, so it is important to divide by its length to first obtain a unit vector. That is,

$$|(1,2)| = \sqrt{1^2+2^2} = \sqrt{5},$$

and so a unit vector pointing in the same direction as $(1, 2)$ is

$$u = \frac{1}{\sqrt{5}}(1,2).$$

Then, the directional derivative we are looking for is

$$D_u\left(\frac{\pi}{2}, \pi\right) = \nabla f\left(\frac{\pi}{2}, \pi\right)\cdot u = \frac{1}{\sqrt{5}}(-1,-1)\cdot(1,2) = \frac{1}{\sqrt{5}}(-1-2) = -\frac{3}{\sqrt{5}}.$$