TUTORIAL 3 SOLUTIONS

MA1132: ADVANCED CALCULUS, HILARY 2017

(1) Suppose that a particle travels with velocity function given for $t > 0$ by

$$v(t) = (t^2, \sqrt{2} \cdot t \log t, (\log t)^2),$$

and that at $t = 1$, the position of the particle is $r(1) = (1, 0, 3)$. Find the following.

(a) The position function $r(t)$ for the particle.
(b) The the distance travelled by the particle from $t = 1$ to $t = 2$.
(c) The acceleration function $a(t)$.

Solution:

a). We have to find an antiderivative of $v(t)$. For this, we compute

$$\int t^2 dt = \frac{t^3}{3} + C,$$

$$\int t \log t dt = \frac{t^2 \log t}{2} - \frac{1}{2} \int t dt = \frac{t^2}{2} \log t - \frac{t^2}{4} + C,$$

(where we used partial integration with $u = \log t$, $du = dt/t$, $v = t^2/2$, $dv = tdt$)

$$\int \log^2 t dt = t \log^2 t - 2 \int \log t dt$$

(where we used partial integration with $u = \log^2 t$, $du = 2 \log t dt/t$, $v = t$, $dv = dt$) which equals

$$t \log^2 t - 2t \log t + 2 \int dt = t \log^2 t - 2t \log t + 2t + C$$

(where we used partial integration with $u = \log t$, $du = dt/t$, $v = t$, $dv = dt$). Thus,

$$r(t) = \left(\frac{t^3}{3}, \frac{\sqrt{2}t^2}{2} \log t - \frac{\sqrt{2}t^2}{4}, t \log^2 t - 2t \log t + 2t \right) + C,$$

where C is now a vector. We can solve for C by plugging in to find

$$r(1) = \left(\frac{1}{3}, -\frac{\sqrt{2}}{4}, 2 \right) + C = (1, 0, 3),$$

so that $C = \left(\frac{2}{3}, \frac{\sqrt{2}}{4}, 1\right)$ and

$$r(t) = \left(\frac{t^3+2}{3}, \frac{\sqrt{2}t^2}{2}\log t - \frac{\sqrt{2}}{4}(t^2-1), t\log^2 t - 2t\log t + 2t + 1\right).$$

b). We need to integrate $|v(t)|$ from $t = 1$ to $t = 2$. Note that

$$|v(t)| = \sqrt{t^4 + 2t^2\log^2 t + \log^4 t} = \sqrt{(t^2+\log^2 t)^2} = \left|t^2 + \log^2 t\right| = t^2 + \log^2 t,$$

where we have used the fact that $t^2 + \log^2 t \geq 0$ on the interval $[1, 2]$. Hence, the distance travelled is

$$\int_1^2 (t^2 + \log^2 t)dt = \left[\frac{t^3}{3} + t\log^2 t - 2t\log t + 2t\right]_1^2$$

$$= (8/3 + 2\log^2 2 - 4\log 2 + 4) - (1/3 + 2) = \frac{13}{3} + 2\log 2(\log 2 - 2).$$

c). The acceleration is

$$a(t) = v'(t) = (2t, \sqrt{2}\log t + \sqrt{2}, 2\log t/t).$$

(2) Determine whether or not the following limit exists, and if it does, find its value:

$$\lim_{(x,y)\to(0,0)} \frac{x + 2\sin y}{x + y}.$$

Solution: The limit does not exist, as we can parameterize two different paths towards the point $(0, 0)$ by $(t, 0)$ and $(0, t)$ both giving the desired point when $t = 0$. The limit towards $t = 0$ on these two curves, are, respectively

$$\lim_{t\to 0} \frac{t}{t} = \lim_{t\to 0} 1 = 1.$$

and

$$\lim_{t\to 0} \frac{2\sin t}{t} = \lim_{t\to 0} \frac{2\cos t}{1} = 2$$

(using L'Hospital's rule). As these two limits approaching the point $(0, 0)$ aren't equal, the limit in question does not exist.

(3) Sketch the domains of the following functions and determine whether they are open sets or not.

(a)

$$f(x, y) = \log\left(1 - \sqrt{x^2 - 4x + y^2 + 4}\right).$$

(b)

$$f(x, y) = \frac{x + \sin y}{y + \cos x}.$$

Solution:

a). The domain is the set where the function inside of the logarithm is positive, that is, where

$$1 - \sqrt{x^2 - 4x + y^2 + 4} > 0,$$

or, equivalently,

$$\sqrt{x^2 - 4x + y^2 + 4} < 1,$$

which is the same as

$$(x - 2)^2 + y^2 < 1.$$

This is just the interior of a circle of radius 1 with center at $(2, 0)$ (without the boundary of the circle included). This is an open set.

b). The only "problem points" are those where the denominator vanishes, which occurs when $y = -\cos(x)$. That is, the domain is the set of all points in the $x - y$ plane which are not on the graph of the curve $y = -\cos x$. This is also an open set.