TUTORIAL 2 SOLUTIONS

MA1132: ADVANCED CALCULUS, HILARY 2017

(1) Consider the parametric curve

$$\begin{cases} x(t) = \cos(t) \\ y(t) = \sin(t) \\ z(t) = \frac{2}{3}t^{\frac{3}{2}}. \end{cases}$$

(a) Using the base point $t_0 = 0$, find, as a function of t, the arc length s of the curve from 0 to t.

(b) Write the arc length parameterization of the curve above.

(c) Find the coordinates of the point on the curve which is an arc length distance of $\frac{14}{3}$ away from the point at $t = 0$ (in the direction of the orientation induced by the parameterization above).

Solution:

a). Setting $r(t) = (\cos(t), \sin(t), \frac{2}{3}t^{\frac{3}{2}})$, we compute

$$r'(t) = (-\sin(t), \cos(t), t^{\frac{1}{2}}).$$

Thus,

$$|r'(t)| = \sqrt{\sin^2(t) + \cos^2(t) + t} = \sqrt{t+1}$$

The arc length from 0 to t is given by

$$s = \int_0^t \left|\frac{dr}{du}\right| du = \int_0^t \sqrt{u+1}du = \frac{2}{3}\left((t+1)^{\frac{3}{2}} - 1\right).$$

b). Solving for s in the last expression gives

$$t = \left(\frac{3s}{2} + 1\right)^{\frac{2}{3}} - 1.$$

Thus, the arc length parameterization of the curve is obtained by substituting the last expression in for t in the original parameterization, yielding

$$\begin{cases} x(s) = \cos\left(\left(\frac{3s}{2} + 1\right)^{\frac{2}{3}} - 1\right) \\ y(s) = \sin\left(\left(\frac{3s}{2} + 1\right)^{\frac{2}{3}} - 1\right) \\ z(s) = \frac{2}{3}\left(\left(\frac{3s}{2} + 1\right)^{\frac{2}{3}} - 1\right)^{\frac{3}{2}}. \end{cases}$$

c) When $s = 14/3$, then $t = \left(\frac{3}{2} \cdot \frac{14}{3} + 1\right)^{\frac{2}{3}} - 1 = 8^{\frac{2}{3}} - 1 = 3$. Thus,

$$(x, y, z) = \left(\cos(3), \sin(3), 2\sqrt{3}\right).$$

(2) In this problem, you will prove the important *Frenet-Serret formulas*, which describe how the TNB frame of unit tangent, unit normal, and binormal vectors of a curve in $\mathbb{R}^3$ change as you move along the curve. These state the following (using arc length parameterization):

$$\frac{dT}{ds} = \kappa N,$$
$$\frac{dN}{ds} = -\kappa T + \tau B,$$
$$\frac{dB}{ds} = -\tau N.$$

Here, τ is the *torsion* (which we briefly saw on the homework), and κ is the curvature of the curve. More succinctly, we can use matrix multiplication notation to write (where $'$ denotes differentiation with respect to s)

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

(a) Show the first equation, namely

$$\frac{dT}{ds} = \kappa N,$$

using the definition

$$N = \frac{\frac{dT}{dt}}{\left|\frac{dT}{dt}\right|},$$

and the following two facts from class:

$$\frac{ds}{dt} = \left|\frac{dr}{dt}\right|,$$

$$\kappa = \frac{\left|\frac{dT}{dt}\right|}{\left|\frac{dr}{dt}\right|}.$$

(b) Show that $\frac{dB}{ds}$ is perpendicular to B. Now show that $\frac{dB}{ds}$ is also perpendicular to T (hint: recall that B is perpendicular to both T and N by its definition as a cross product $B = T \times N$ and differentiate the equation $0 = B \cdot T$). Conclude that

$$\frac{dB}{ds} = -\tau N,$$

where τ is a scalar-valued function (the minus sign is unimportant, and only there for historical reasons).

(c) Show, by differentiating the equation $N = B \times T$ that for the same function τ you defined by the equation in b), we have

$$\frac{dN}{ds} = -\kappa T + \tau B.$$

Solution:

a) We use the chain rule to find

$$\frac{dT}{dt} = \frac{dT}{ds}\frac{ds}{dt},$$

so that by using the formulas above we find

$$\frac{dT}{ds} = \frac{\frac{dT}{dt}}{\frac{ds}{dt}} = \frac{\left|\frac{dT}{dt}\right|}{\left|\frac{dr}{dt}\right|}N = \kappa N.$$

b) Since $|B| = 1$ for all s, using a basic theorem from class, dB/ds is always perpendicular to B. Since B is perpendicular to both T and N, both dot products $B \cdot T$ and $B \cdot N$ are equal to zero. Differentiating $0 = B \cdot T$, we find, using the product rule for dot products, that

$$0 = B \cdot \frac{dT}{ds} + \frac{dB}{ds} \cdot T,$$

which by the first Frenet-Serret formula is equal to

$$0 = \kappa B \cdot N + \frac{dB}{ds} \cdot T = \frac{dB}{ds} \cdot T.$$

Thus, dB/ds is perpendicular to both T and B, implying that it is a multiple of N. We then simply choose to call this multiple $-\tau$.

c) We use the product rule for taking derivatives of cross products to find

$$\frac{dN}{ds} = \frac{dB}{ds} \times T + B \times \frac{dT}{ds} = -\tau(N \times T) + \kappa(B \times N),$$

where in the last equality we used parts a) and b). Since the cross product is an anticommutative operation, $N \times T = -(T \times N) = -B$. Moreover, $B \times N$ is perpendicular to both B and N, and has length 1, so that it is $\pm T$. A check of the right-hand rule shows that in fact $B \times N = -T$. Plugging these last two facts into the last displayed equation shows that

$$\frac{dN}{ds} = \tau B - \kappa T,$$

which is equivalent to the claim.

(3) Find the curvature of the plane curve parameterized by $r(t) = (t, t^2)$ at the point when $t = 2$.

Solution: We will use the formula

$$\kappa(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}.$$

We are using a cross product, so we must embed this curve in three dimensions, and so will use the parameterization $r(t) = (t, t^2, 0)$ for the same curve (using the same letter r). We then find

$$r'(t) = (1, 2t, 0),$$
$$r''(t) = (0, 2, 0),$$
$$r'(t) \times r''(t) = (0, 0, 2).$$

Thus, $|r'(t) \times r''(t)| = 2$ and $|r'| = \sqrt{1 + 4t^2}$. Therefore, for any t,

$$\kappa(t) = \frac{2}{(1 + 4t^2)^{\frac{3}{2}}}.$$

Plugging in $t = 2$ yields

$$\kappa(2) = \frac{2}{17^{\frac{3}{2}}}.$$