TUTORIAL 1, SOLUTIONS

MA1132: ADVANCED CALCULUS, HILARY 2017

N.B. Annotated pictures of the geometric objects in the solutions can be find at the end of these solutions.

(1) (a) Find the equation of the plane passing through the points $(1,1,3)$, $(0,0,-2)$, and $(2,2,1)$.
(b) Find parametric equations describing the curve of intersection of this plane with the surface given by $z = x^2 + y^2$.
(c) What type of geometric object is the curve you found in b)?

Solution:

a) Call the points $A = (1,1,3)$, $B = (0,0,-2)$, and $C = (2,2,1)$. Then we have $\overrightarrow{AB} = (-1,-1,-5)$, $\overrightarrow{AC} = (1,1,-2)$. Computing the cross product, we find $\overrightarrow{AB} \times \overrightarrow{AC} = (7,-7,0) = 7(1,-1,0)$. Thus, we may take as a normal vector $n = (1,-1,0)$ and as a point on the plane B, giving the equation

$$(x-0) - (y-0) + 0 \cdot (z+2) = x - y = 0,$$

or $x = y$.

b). We want to find the intersection of $x = y$ and $z = x^2+y^2$. We parameterize this intersection by setting $x = t$, so that $y = t$ as well, and by plugging into the last equation, we find the parameterization

$$\begin{cases} x = t \\ y = t \\ z = t^2 + t^2 = 2t^2. \end{cases}$$

c). The equations we have just described determine a parabola in the plane $x = y$.

(2) Show that the graph of the vector-valued function

$$\vec{r}(t) = t\cos t\vec{i} + t\sin t\vec{j} + t\vec{k}$$

lies on the double-cone $x^2 + y^2 = z^2$.

Solution:

The graph of this function consists of points parameterized by the equations

$$\begin{cases} x = t\cos t \\ y = t\sin t \\ z = t. \end{cases}$$

We have to show that $x(t)^2 + y(t)^2 = z(t)^2$. Indeed,

$$(t\cos t)^2 + (t\sin t)^2 = t^2(\sin^2 t + \cos^2 t) = t^2 = z(t)^2.$$

(3) Consider the surface given parametrically in terms of parameters $u, v \in [0, 2\pi)$ by

$$\begin{cases} x = (2+\cos v)\cos u \\ y = (2+\cos v)\sin u \\ z = \sin v. \end{cases}$$

(a) The intersection of this surface with the plane $y = 0$ is a union of two curves. Describe what these two curves are by finding (non-parametric) equations for them in a form which makes the geometric interpretation of these two curves clear.

(b) Now consider the intersection of the same surface with the plane $z = 0$ and find non-parametric equations for the curves in this intersection, and describe the objects you find.

Solution:

a). The surface, as we saw in class, is a torus. As $(2 + \cos v)$ never equals 0, if $y = 0$, then $\sin u = 0$, and so $\cos u = \pm 1$. Thus, our system of equations reduces to the system of parametric equations

$$\begin{cases} x = \pm(2+\cos v) \\ y = 0 \\ z = \sin v. \end{cases}$$

In each case, this is a circle. We can find these circles by eliminating the variables as follows. We must somehow use the fact that $\sin^2 v + \cos^2 v = 1$. We solve for $\cos v$ to find

$$\cos v = \mp x - 2$$

Thus,

$$1 = \sin^2 v + \cos^2 v = z^2 + (x \pm 2)^2,$$

and our two ellipses are thus those lying in the xz-plane satisfying

$$(x \pm 2)^2 + z^2 = 1.$$

That is, they are circles of radius 1 centered at $(\mp 2, 0, 0)$.

b). If $z = 0$, then $\sin v = 0$, and so $\cos v = \pm 1$, and so we have the two curves in the xy-plane with equations

$$\begin{cases} x = 3\cos u \\ y = 3\sin u, \end{cases}$$

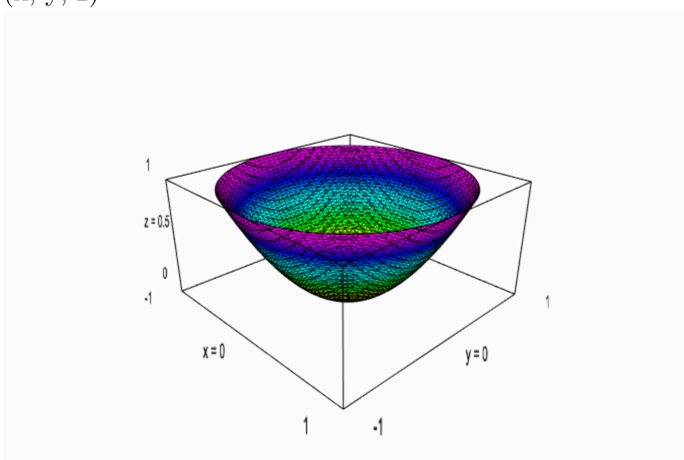
$$\begin{cases} x = \cos u \\ y = \sin u, \end{cases}$$

These are circles centered at the origin with radii 3 and 1, respectively, and are described by the equations $x^2 + y^2 = 9$ and $x^2 + y^2 = 1$.

```
## Problem 1
## The surface in b is a paraboloid, which is obtained by rotating a\
    parabola around the z-axis.
```

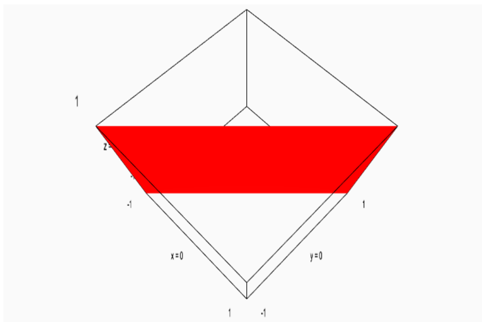
```
var('x,y,z')
cm = colormaps.gist_rainbow
implicit_plot3d(x^2+y^2==z, (x,-1,1), (y,-1,1), (z,0,1),mesh=true,\
    color=(z,cm))
```

(x, y, z)

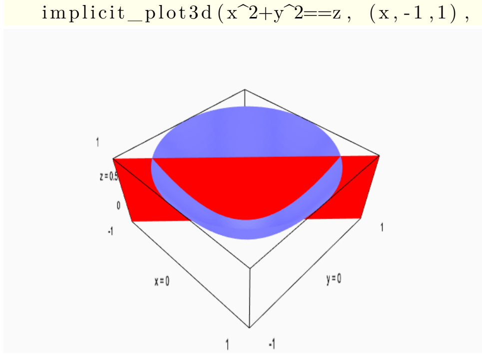



```
### We will intersect the paraboloid with the plane x=y, viewed from\
    this angle.
```

```
implicit_plot3d(x==y, (x,-1,1), (y,-1,1), (z,-1,1),color='red')
```



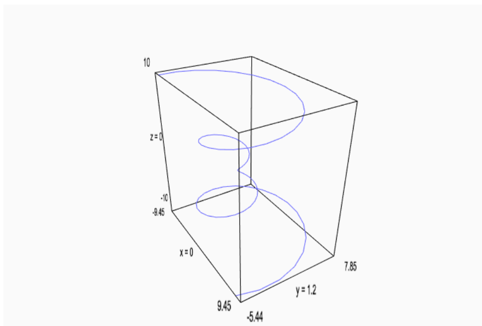

```
### The intersection is a parabola.
show(implicit_plot3d(x==y, (x,-1,1), (y,-1,1), (z,0,1),color='red')+\
    implicit_plot3d(x^2+y^2==z, (x,-1,1), (y,-1,1), (z,0,1)))
```



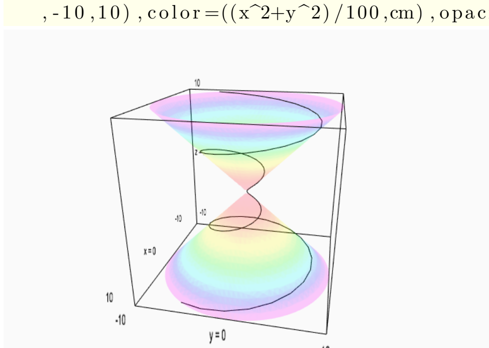

```
### Problem 2: First we draw the curve sketched by the vector-valued\
     function r(t).
```

```
var('t')
parametric_plot3d((t*cos(t),t*sin(t),t), (t,-10,10))
```

t



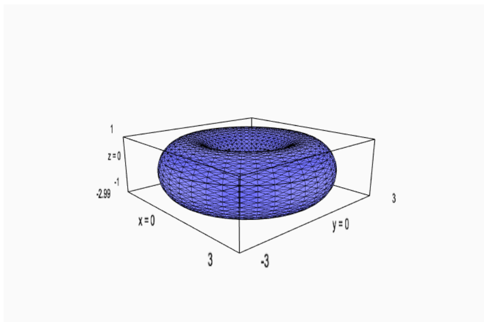

```
### And now we graph to see it lying on the surface of this double \
    cone. If you use SAGE and plug in these functions, you can rotate\
     the picture around to see it more clearly!

show(parametric_plot3d((t*cos(t),t*sin(t),t), (t,-10,10),color='\
    black')+implicit_plot3d(x^2+y^2==z^2, (x,-10,10), (y,-10,10), (z\
    ,-10,10),color=((x^2+y^2)/100,cm),opacity=0.2))
```




```
### Problem 3: We first draw the torus.
```

```
var('u,v')
parametric_plot3d(((2+cos(v))*cos(u), (2+cos(v))*sin(u), sin(v)), (u\
    ,0,2*pi), (v,0,2*pi), mesh=True)
```

(u, v)




```
### A picture of the two planes intersecting the torus. One plane \
    intersects the torus at two concentric circles, while the other \
    intersects it at two circles of the same radius translated from \
    one another.
```

```
show(parametric_plot3d(((2+cos(v))*cos(u), (2+cos(v))*sin(u), sin(v)\
    ), (u,0,2*pi), (v,0,2*pi), opacity=0.2)+implicit_plot3d(y==0, (x\
    ,-3,3), (y,-3,3), (z,-3,3),color='red')+implicit_plot3d(z==0, (x\
    ,-3,3), (y,-3,3), (z,-3,3),color='green'))
```