HOMEWORK 9 SOLUTIONS

MA1132: ADVANCED CALCULUS, HILARY 2017

(1) Find the surface area of the piece of the plane $z = x + y$ lying inside the cylinder $(x-2)^2 + (y-3)^2 = 1$.

Solution:

We want to find the surface area of the piece of the plane lying over R, where R is the circle in the x-y plane of radius 1 centered at $(2,3)$. The partial derivatives of $f(x,y) = x + y$ are $f_x = f_y = 1$, and so this surface area is given by

$$\iint_R \sqrt{f_x^2 + f_y^2 + 1}dA = \sqrt{3}\iint_R dA,$$

which is $\sqrt{3}$ times the area of R. Since this area is π, the surface area of the corresponding piece of the plane is $\pi\sqrt{3}$.

(2) Gabriel's horn is a famous shape obtained by rotating the area under the curve $y = 1/x$ in the x-y plane from $x = 1$ to ∞ around the x-axis. Find parametric equations for this surface, and find an integral expression for the surface area of the "truncated" horn from $x = 1$ to $x = a$. Conclude, by using a comparison with a divergent integral, that this horn has infinite surface area.

Solution: Using the method sketched in class, we can give parametric equations for the horn by first setting $x = u$, and then $y = f(u)\cos(v)$, $z = f(u)\sin(v)$, where in this case $f(x) = 1/x$. This gives that the horn is the graph of the multivariable, vector-valued function

$$r(u,v) = \left(u, \frac{\cos v}{u}, \frac{\sin v}{u}\right).$$

We now compute the partial derivatives of r (recall that this is done componentwise):

$$\frac{\partial r}{\partial u} = \left(1, -\frac{\cos v}{u^2}, -\frac{\sin v}{u^2}\right),$$

$$\frac{\partial r}{\partial v} = \left(0, -\frac{\sin v}{u}, \frac{\cos v}{u}\right).$$

We then compute

$$\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} = \left(-\frac{1}{u^3}, -\frac{\cos v}{u}, -\frac{\sin v}{u}\right),$$

and so

$$\left|\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}\right| = \sqrt{\frac{1}{u^6} + \frac{1}{u^2}} = \sqrt{\frac{u^4+1}{u^6}}.$$

Thus, the surface area of the horn from $x = 1$ to $x = a$ is the integral over the region $[1, a] \times [0, 2\pi]$ in the u-v plane of this quantity, that is,

$$\int_0^{2\pi} \int_1^a \sqrt{\frac{u^4+1}{u^6}} du dv = 2\pi \int_1^a \sqrt{\frac{u^4+1}{u^6}} du.$$

For very large u, the term u^4/u^6 dominates $1/u^6$, so the integrand is approximately $1/u$ for large u. Thus, we can see the reason why this last integral diverges for $a = \infty$; it is an integral of a function that is eventually very nearly $1/u$, which we know has divergent integral on the interval $[1, \infty)$. We can be more explicit by noting that

$$\sqrt{\frac{u^4+1}{u^6}} > \frac{1}{u},$$

so that we have the corresponding inequality of integrals:

$$\int_1^a \sqrt{\frac{u^4+1}{u^6}} du > \int_1^a \frac{du}{u} = \log a.$$

As $a \to \infty$, this thus diverges, and hence so does the surface area of Gabriel's horn.

(3) Evaluate the iterated integral

$$\int_{-1}^1 \int_{-x}^x \int_0^{x^2+z} x \sin(x^7) y^2 dy dz dx.$$

(Hint: For the final integral over x, what do you notice about the integrand?)

Solution: We evaluate

$$\begin{aligned}\int_{-1}^{1}\int_{-x}^{x}\int_{0}^{x^2+z} x\sin(x^7)y^2\,dydzdx &= \int_{-1}^{1}\int_{-x}^{x} x\sin(x^7)\left[\frac{y^3}{3}\right]_{y=0}^{x^2+z} dzdx\\
&= \int_{-1}^{1}\int_{-x}^{x} \frac{x\sin(x^7)}{3}(x^2+z)^3\,dzdx\\
&= \int_{-1}^{1}\int_{-x}^{x} \frac{x\sin(x^7)}{3}(x^6+3x^4z+3x^2z^2+z^3)\,dzdx\\
&= \int_{-1}^{1} \frac{x\sin(x^7)}{3}\left[x^6z+\frac{3x^4z^2}{2}+x^2z^3+\frac{z^4}{4}\right]_{z=-x}^{x} dx\\
&= \int_{-1}^{1} \frac{x\sin(x^7)}{3}(2x^7+2x^5)\,dx\\
&= \frac{2}{3}\int_{-1}^{1}(x^8+x^6)\sin(x^7)\,dx.\end{aligned}$$

Now we don't need to (and you probably wouldn't be able to) find the explicit antiderivative of this function. Note that the integrand is an odd function (meaning that $f(-x) = -f(x)$), and since it is integrated over the symmetric interval $[-1, 1]$ about the origin, the integral is 0. Thus,

$$\int_{-1}^{1}\int_{-x}^{x}\int_{0}^{x^2+z} x\sin(x^7)y^2\,dydzdx = 0.$$

(4) Find the volume of the region between the paraboloid $z = x^2 + y^2$ and the x-y plane above the annular region S lying between the concentric circles of radii 1 and 2 centered at the origin.

Solution: If we call this three-dimensional region R, then the volume of R is given by the triple integral $\iiint_R dV$. This is a simple x-y solid above S, where for fixed x, y, z ranges from 0 to $x^2 + y^2$. Thus,

$$V = \iiint_R dV = \iint_S \left(\int_0^{x^2+y^2} dz\right) dA = \iint_S (x^2+y^2)\,dA.$$

We use polar coordinates to evaluate this last double integral. In polar coordinates, $x^2 + y^2 = r^2$, $dA = r\,dr\,d\vartheta$, and S is the simple polar region where ϑ ranges from 0 to 2π and for fixed ϑ, r ranges from 1 to 2. Thus,

$$V = \iint_S (x^2+y^2)\,dA = \int_0^{2\pi}\int_1^2 r^3\,dr\,d\vartheta = 2\pi\left[\frac{r^4}{4}\right]_{r=1}^{2} = \frac{15\pi}{2}.$$