HOMEWORK 8 SOLUTIONS

MA1132: ADVANCED CALCULUS, HILARY 2017

(1) Compute the double integral

$$\iint_R \sqrt{y}dA,$$

where R is the region between the curves $y = \sqrt{x}$ and $y = x^2$. (Hint: break the region R into two pieces.)

Solution: The curves intersect at $x = 0, 1$, and in the interval $[0, 1]$ the curve $y = \sqrt{x}$ always lies above the curve $y = x^2$ (this follows from the fact that they don't intersect on that interval and noting that $\sqrt{x} > x^2$ in at least one point in the interval by just plugging a point in). Thus, R is a type I region with x ranging from 0 to 1, and wherein any vertical line through a fixed x runs from $y = x^2$ to $y = \sqrt{x}$. Thus, the double integral above can be expressed as the iterated integral

$$\begin{aligned}\iint_R \sqrt{y}dA &= \int_0^1 \int_{x^2}^{\sqrt{x}} \sqrt{y}dydx \\ &= \int_0^1 \left[\frac{2y^{\frac{3}{2}}}{3}\right]_{y=x^2}^{\sqrt{x}} dx = \int_0^1 \left(\frac{2x^{\frac{3}{4}}}{3} - \frac{2x^3}{3}\right) dx \\ &= \left[\frac{8x^{\frac{7}{4}}}{21} - \frac{x^4}{6}\right]_{x=0}^1 = \frac{3}{14}.\end{aligned}$$

(2) Find the value of

$$\iint_R (x^2 - y)dA$$

where R is the square with vertices $(-1, 0)$, $(1, 0)$, $(0, 1)$, and $(0, -1)$.

Solution: This square is not nice to express as either a type I or a type II region, as one would have to deal with absolute value, or piecewise functions. We thus break this region into two simpler type I regions by splitting the square down the middle line at $x = 0$. Call R_1 the piece of the square to the left of this line, and R_2 the region to the right (including or not including the boundary piece in the middle doesn't change anything; two-dimensional integrals over one-dimensional curves always give zero, just as single integrals zero-dimensional points give zero). These are both type I regions, with x ranging from -1 to 0 in

R_1, and x ranging from 0 to 1 in R_2. In R_1, for fixed vertical lines through x, y ranges from the line from $(-1,0)$ to $(0,-1)$, which is given by $y=-x-1$, to the line from $(-1,0)$ to $(0,1)$, which is given by $y=x+1$. In R_2, for fixed vertical lines through x, y ranges from the line from $(0,-1)$ to $(1,0)$, which is given by $y=x-1$, to the line from $(0,1)$ to $(1,0)$, which is given by $y=-x+1$. Thus,

$$\begin{aligned}\iint_R (x^2-y)dA &= \iint_{R_1}(x^2-y)dA + \iint_{R_2}(x^2-y)dA\\ &= \int_{-1}^{0}\int_{-x-1}^{x+1}(x^2-y)dydx + \int_0^1\int_{x-1}^{-x+1}(x^2-y)dydx\\ &= \int_{-1}^{0}\left[x^2y-\frac{y^2}{2}\right]_{y=-x-1}^{x+1}dx + \int_0^1\left[x^2y-\frac{y^2}{2}\right]_{y=x-1}^{-x+1}dx\\ &= \int_{-1}^{0}(2x^3+2x^2)dx + \int_0^1(-2x^3+2x^2)dx\\ &= \left[\frac{x^4}{2}+\frac{2x^3}{3}\right]_{x=-1}^{0} + \left[-\frac{x^4}{2}+\frac{2x^3}{3}\right]_{x=0}^{1}\\ &= \frac{1}{6}+\frac{1}{6}=\frac{1}{3}.\end{aligned}$$

(3) Let R be the region in the x-y plane bounded by the lines $y=1$, $y=2$, the y-axis, and the curve $y=1/x$. Find the volume lying over R und under the graph of the function $f(x,y)=e^{xy}$.

Solution: Here, R is a type II region, bounded by $y=1$ and $y=2$, and for fixed horizontal lines at height y in R, x ranges from 0 to $1/y$ (solve for x in $y=1/x$). Thus, the volume we are looking for is given by (note that $e^{xy}>0$ on R as e^x is always positive, so this is the volume and not just the net volume)

$$\begin{aligned}\iint_R e^{xy}dA &= \int_1^2\int_0^{\frac{1}{y}} e^{xy}dxdy\\ &= \int_1^2\left[\frac{e^{xy}}{y}\right]_{x=0}^{\frac{1}{y}}dy\\ &= \int_1^2\frac{e-1}{y}dy = (e-1)\int_1^2\frac{dy}{y} = (e-1)[\log y]_{y=1}^2 = (e-1)\log 2.\end{aligned}$$

(4) Use polar coordinates to compute

$$\iint_R xydA$$

where R is the region lying between the concentric circles of radii 1 and 2 centered at the origin and in the first quadrant (this is one quarter of an annulus).

Solution: In polar coordinates, R is a simple polar region. In this case, ϑ ranges from 0 to $\pi/2$, and along any ray of constant angle ϑ, r ranges from 1 to 2. Moreover, $x = r\cos\vartheta$, $y = r\sin\vartheta$, and $dA = rdrd\vartheta$. Thus, we have

$$\begin{aligned}\iint_R xydA &= \int_0^{\frac{\pi}{2}}\int_1^2 r^3\cos\vartheta\sin\vartheta drd\vartheta \\ &= \int_0^{\frac{\pi}{2}}\left[\frac{r^4}{4}\right]_{r=1}^2 \cos\vartheta\sin\vartheta d\vartheta \\ &= \frac{15}{4}\int_0^{\frac{\pi}{2}}\cos\vartheta\sin\vartheta = \frac{15}{4}\left[-\frac{1}{2}\cos^2\vartheta\right]_{\vartheta=0}^{\frac{\pi}{2}} \\ &= \frac{15}{8}.\end{aligned}$$

(5) Compute the value of

$$\int_{-1}^1\int_0^{\sqrt{1-x^2}}\cos(x^2+y^2)dydx$$

by switching to polar coordinates.

Solution: We first try to identify this as a double integral over a type I region (since the integration order is $dydx$). The equation $y = \sqrt{1-x^2}$ implies that $x^2 + y^2 = 1$, and so we recognize this graph between $x = -1$ and $x = 1$ as the upper half of a semi-circle of radius 1 centered at the origin. Thus,

$$\int_{-1}^1\int_0^{\sqrt{1-x^2}}\cos(x^2+y^2)dydx = \iint_R \cos(x^2+y^2)dA,$$

where R is the upper half of the closed unit disc of radius 1 centered at the origin. Thus, this is a simple polar region with ϑ ranging from 0 to π and for fixed θ we have r ranging from 0 to 1. Noting that $x^2 + y^2 = r^2$ and $dA = rdrd\vartheta$, we find

$$\iint_R \cos(x^2+y^2)dA = \int_0^\pi\int_0^1\cos(r^2)rdrd\vartheta = \int_0^\pi\left[\frac{\sin(r^2)}{2}\right]_{r=0}^1 d\vartheta = \frac{\sin(1)}{2}\int_0^\pi d\vartheta = \frac{\sin(1)\pi}{2}.$$