HOMEWORK 5 SOLUTIONS

MA1132: ADVANCED CALCULUS, HILARY 2017

(1) Consider the function $z = f(x,y) = x\log(xy) - \sqrt{x^2+y^2}$ with $x = t^2+1$, $y = t-1$. Find $\frac{dz}{dt}$ by using the chain rule.

Solution: We compute

$$\frac{\partial f}{\partial x} = \log(xy) + 1 - \frac{x}{\sqrt{x^2+y^2}},$$

$$\frac{\partial f}{\partial y} = \frac{x}{y} - \frac{y}{\sqrt{x^2+y^2}},$$

$$\frac{dx}{dt} = 2t,$$

$$\frac{dy}{dt} = 1.$$

$$x^2+y^2 = t^4+3t^2-2t+2,$$

$$xy = t^3-t^2+t-1,$$

and so

$$\frac{dz}{dt} = \left(\log(xy) + 1 - \frac{x}{\sqrt{x^2+y^2}}\right)\cdot(2t) + \left(\frac{x}{y} - \frac{x}{\sqrt{x^2+y^2}}\right)\cdot(1)$$

$$= 2\left(\log(t^3-t^2+t-1) + 1 - \frac{t^2+1}{\sqrt{t^4+3t^2-2t+2}}\right)t + \left(\frac{t^2+1}{t-1} - \frac{t-1}{\sqrt{t^4+3t^2-2t+2}}\right).$$

(2) Suppose that $w = f(x,y,z) = xy^{\frac{1}{2}} + \sin\left(\frac{x}{y}\right)\tan z - z^2x^3$ and $x = 2r+s$, $y = st$, $z = r-t$. Find $\frac{\partial w}{\partial r}$.

Solution:

Using a tree diagram, the chain rule for this situation becomes

$$\frac{\partial w}{\partial r} = f_x\frac{\partial x}{\partial r} + f_y\frac{\partial y}{\partial r} + f_z\frac{\partial z}{\partial r}.$$

Now, we directly compute that

$$f_x = y^{\frac{1}{2}} + \frac{\cos\left(\frac{x}{y}\right)}{y}\tan z - 3z^2x^2 = \sqrt{st} + \frac{\cos\left(\frac{2r+s}{st}\right)}{st}\tan(r-t) - 3(r-t)^2(2r+s)^2,$$

$$f_y = \frac{x}{2\sqrt{y}} - \frac{x\cos\left(\frac{x}{y}\right)}{y^2}\tan z = \frac{2r+s}{2\sqrt{st}} - \frac{(2r+s)\cos\left(\frac{2r+s}{st}\right)}{s^2t^2}\tan(r-t),$$

$$f_z = \sec^2 z \sin\left(\frac{x}{y}\right) - 2zx^3 = \sec^2(r-t)\sin\left(\frac{2r+s}{st}\right) - 2(r-t)(2r+s)^3,$$

$$\frac{\partial x}{\partial r} = 2,$$

$$\frac{\partial y}{\partial r} = 0,$$

$$\frac{\partial z}{\partial r} = 1,$$

and so

$$\frac{\partial w}{\partial r} = 2\left(\sqrt{st} + \frac{\cos\left(\frac{2r+s}{st}\right)}{st}\tan(r-t) - 3(r-t)^2(2r+s)^2\right) + \left(\sec^2(r-t)\sin\left(\frac{2r+s}{st}\right) - 2(r-t)(2r+s)^3\right).$$

(3) Find $\left.\frac{\partial^2 f}{\partial\vartheta^2}\right|_{\vartheta=\frac{\pi}{2},\ r=\sqrt{3}}$ for $f(x,y) = xy + y^2$, $x = r\cos\vartheta$, $y = r\sin\vartheta$.

Solution: We start with the first derivative with respect to ϑ. Using

$$\frac{\partial f}{\partial\vartheta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial\vartheta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial\vartheta},$$

$$f_x = y,$$

$$f_y = x + 2y,$$

$$\frac{\partial x}{\partial\vartheta} = -r\sin\vartheta,$$

$$\frac{\partial y}{\partial\vartheta} = r\cos\vartheta,$$

we have

$$\begin{aligned}\frac{\partial f}{\partial\vartheta} &= -r\sin(\vartheta)y + r\cos\vartheta(x+2y) \\ &= -r^2\sin^2\vartheta + r\cos(\vartheta)\left(r\cos\vartheta + 2r\sin\vartheta\right) \\ &= r^2\left(\cos^2\vartheta - \sin^2\vartheta + 2\sin\vartheta\cos\vartheta\right).\end{aligned}$$

Thus, differentiating one more time with respect to ϑ, we find

$$\frac{\partial^2 f}{\partial\vartheta^2} = \frac{\partial}{\partial\vartheta}\left(r^2\left(\cos^2\vartheta - \sin^2\vartheta + 2\sin\vartheta\cos\vartheta\right)\right) = r^2\left(-2\sin\vartheta\cos\vartheta - 2\sin\vartheta\cos\vartheta + 2\cos^2\vartheta - 2\sin^2\vartheta\right).$$

(Note that if you use the very general formula for second partial derivatives with respect to ϑ on a surface expressed in polar coordinates which we gave in class, then you will get the same answer, but it is much simpler in specific examples!) Plugging in $\vartheta = \pi/2$, $r = \sqrt{3}$, we obtain

$$3 \cdot (-2) = -6.$$

(4) Find the directional derivative of $f(x, y, z) = \frac{x+y^2}{x-y^3z}$ in the direction of the line in the plane $z = 0$ which makes an angle of $\pi/3$ with the x-axis (in the direction of increasing x) as well as in the direction of the vector $(1, 2, 3)$ at the point $(1, -1, 1)$.

Solution:

We have

$$f_x = -\frac{y^2(yz+1)}{(x-y^3z)^2},$$

$$f_y = \frac{y(y^3z + 3xyz + 2x)}{(x-y^3z)^2},$$

$$f_z = \frac{(y^2+x)y^3}{(x-y^3z)^2},$$

$$\nabla f = (f_x, f_y, f_z).$$

In any direction pointing in the direction of a unit vector u, the directional derivative $D_uf(1, -1, 1)$ is equal to $\nabla f(1, -1, 1) \cdot u = (0, 1/2, -1/2) \cdot u$. In the first case, a unit vector pointing in the direction of increasing x making an angle of $\pi/3$ with the x-axis in the $x - y$ plane is

$$(\cos(\pi/3), \sin(\pi/3)),$$

and so the unit vector pointing in the specified direction is

$$u = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right),$$

which gives

$$D_uf(1, -1, 1) = (0, 1/2, -1/2) \cdot \left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right) = \frac{\sqrt{3}}{4}.$$

In the second case we need a **unit** vector which points in the same direction as $(1, 2, 3)$. To find this, we just divide by the length of $(1, 2, 3)$, which is $\sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$. That is,

$$u = \frac{(1, 2, 3)}{\sqrt{14}}$$

and so

$$D_uf(1, -1, 1) = \frac{1}{\sqrt{14}}(0, 1/2, -1/2) \cdot (1, 2, 3) = -\frac{1}{2\sqrt{14}}.$$

(5) Find a unit vector pointing in the direction in which f increases the fastest at the point $(1,1)$, when

$$f(x,y) = \frac{x}{y} - \frac{y^{\frac{3}{2}}}{x}.$$

How fast is f increasing in this direction?

Solution: We compute

$$f_x = \frac{x^2 + y^{\frac{5}{2}}}{x^2 y},$$

$$f_y = -\frac{3y^{\frac{5}{2}} + 2x^2}{2xy^2}.$$

At $(1,1)$, we plug into the partial derivatives above to get

$$\nabla f(1,1) = (2, -5/2),$$

$$|\nabla f(1,1)| = \frac{\sqrt{41}}{2}.$$

The function f increases the fastest in the direction of $\nabla f(1,1)$, a unit vector in the direction of which is

$$u = \frac{\nabla f(1,1)}{|\nabla f(1,1)|} = \left(\frac{4}{\sqrt{41}}, -\frac{5}{\sqrt{41}}\right).$$

The rate of increase in this direction is

$$D_u f(1,1) = |\nabla f(1,1)| = \frac{\sqrt{41}}{2}.$$