HOMEWORK 4 SOLUTIONS

MA1132: ADVANCED CALCULUS, HILARY 2017

(1) In class, we considered the piecewise function

$$f(x,y) = \begin{cases} \frac{-xy}{x^2+y^2} & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0). \end{cases}$$

Compute the partial derivatives f_x and f_y and show that they exit everywhere. Conclude that this function has both first-order partial derivatives existing everywhere but that the original function isn't continuous everywhere (and hence, the function isn't differentiable).

Solution: We must consider two cases, when (x, y) isn't or is equal to $(0, 0)$. In the first case, we simply apply the quotient rule to compute:

$$f_x = \frac{\partial f}{\partial x} = \frac{-y(x^2+y^2) + 2x \cdot (xy)}{(x^2+y^2)^2} = \frac{x^2y - y^3}{(x^2+y^2)^2}.$$

Similarly, we find (by symmetry, this must be what we get when we flip x and y in the last line).

$$f_y = \frac{y^2x - x^3}{(x^2+y^2)^2}.$$

Both of these are defined away from $(0, 0)$, as the denominator only vanishes at that point. At the origin, we must compute the partial derivatives directly from the definition. For example,

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0,$$

and similarly $f_y(0, 0) = 0$. Hence, f_x and f_y are defined for all $(x, y) \in \mathbb{R}^2$. However, f isn't continuous at $(0, 0)$, as we saw that $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist (since limits along lines approaching the origin from different directions give different answers).

(2) You are standing on a hill shaped like the torus defined by the equation

$$(x^2 + y^2 + z^2 + 3)^2 = 16(x^2 + y^2), \quad z \geq 0,$$

where z points in the upwards direction, x in the eastwards direction, and y in the northwards direction. If you are at the point $(3/2, 0, \sqrt{3}/2)$, then what is the angle of ascent that your path takes if you travel straight east?

Solution: We first find the slope of the torus in the x-direction at this point, that is, $\frac{\partial z}{\partial x}$. We can find this by implicitly differentiating the equation of the torus:

$$\frac{\partial}{\partial x}\left((x^2+y^2+z^2+3)^2\right) = 2(x^2+y^2+z^2+3)\left(2x+2z\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(16(x^2+y^2)\right) = 32x.$$

Plugging in at the point $(3/2, 0, \sqrt{3}/2)$ gives

$$4\cdot 6\cdot\left(\frac{3}{2}+\frac{\sqrt{3}}{2}\frac{\partial z}{\partial x}\Big|_{x=\frac{3}{2},\ y=0,\ \frac{\sqrt{3}}{2}}\right) = 48,$$

and so

$$\frac{\partial z}{\partial x}\Big|_{x=\frac{3}{2},\ y=0,\ \frac{\sqrt{3}}{2}} = \frac{\sqrt{3}}{3}.$$

Thus, the slope of the line of ascent in the x-direction (or eastern direction) is $\sqrt{3}/3$, and so the angle made with the horizontal is $\arctan(\sqrt{3}/3) = \pi/6$, or 30°.

(3) The one-dimensional *wave equation*, which describes the propagation of waves with respect to space and time, is for a function $u(x,t)$, which is meant to describe the shape of the wave at each point in time t, we have

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}.$$

Here, $c > 0$ is a constant depending on the wave. This is an example of a *partial differential equation* (PDE), which often arise naturally in physics. Although such equations are typically very hard to solve, in this case, it is reasonable to do so, and in fact the general solution is given by

$$u(x,t) = F(x-ct) + G(x+ct)$$

for one-variable functions F, G. Show that for any twice-differentiable functions F and G, that this is indeed a solution of the wave equation. Moreover, show that if a solution to the equation satisfies the initial conditions

$$U(x,0) = f(x),$$

$$U_t(x,0) = g(x),$$

then the function

$$U(x,t) = \frac{f(x-ct)+f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds$$

solves the wave equation with the given initial conditions.

Solution: For the first part, using the chain rule we find that

$$u_{tt} = (u_t)_t = (-cF'(x-ct)+cG'(x+ct))_t = c^2F''(x-ct)+c^2G''(x+ct) = c^2(F''(x-ct)+G''(x+ct)) = c^2u_{xx}.$$

For the second part, note by the Fundamental Theorem of Calculus that U is of the form of the first part as

$$U(x,t) = F(x-ct) + G(x+ct)$$

with $G(x) = f(x)/2 + \mathcal{G}(x)/(2c)$ and $F(x) = f(x)/2 - \mathcal{G}(x)/(2c)$, where $\mathcal{G}$ is an antiderivative of g. To check that it satisfies the initial conditions, we plug in

$$U(x,0) = \frac{f(x)+f(x)}{2} + \frac{1}{2c}\int_x^x g(s)ds = f(x),$$

and

$$U_t(x,t) = \frac{-cf'(x-ct)+cf'(x+ct)}{2} + \frac{1}{2}(g(x+ct)+g(x-ct))$$

and so

$$U_t(x,0) = \frac{-cf'(x)+cf'(x)}{2} + \frac{1}{2}(g(x)+g(x)) = g(x).$$

(4) Using a two-variable linear approximation around the point $(x_0, y_0) = (4, 2)$, approximate the value $f(3.9, 2.05)$ where

$$f(x,y) = 3x^2y - \frac{x}{y}.$$

Solution: We first compute

$$f(4,2) = 94$$

and $f_x(4,2) = \left(6xy - \frac{1}{y}\right)\Big|_{x=4,\ y=2} = \frac{95}{2}$, $f_y(4,2) = \left(3x^2 + \frac{x}{y^2}\right)\Big|_{x=4,\ y=2} = 49$. Thus, with $(x,y) = (3.9, 2.05)$, we have

$$f(x,y) \approx L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) = 94 - \frac{95}{20} + \frac{49}{20} = 91.7.$$

The "real" answer is in fact $f(x,y) = 91.64$, which is off with only a relative error of about 0.07%.

(5) Find the equation of the tangent plane to the surface

$$x^2 + y^2 - 4 = z$$

at $(2,-2,0)$.

Solution: The equation for the linear approximation given in the solution to the last problem also gives the equation for the tangent plane. In this situation, $\frac{\partial z}{\partial x} = 2x$, and $\frac{\partial z}{\partial y} = 2y$. At the point $(x_0, y_0) = (2, -2)$, these values become 4 and -4, respectively. Moreover, we of course have that $z = 0$ when $x = 2$, $y = 2$. Thus, the equation of the plane is

$$z = 0 + 4(x-x_0) - 4(y-y_0) = 4(x-2) - 4(y+2) = 4x - 4y - 16.$$

Equivalently,

$$4x - 4y - z = 16.$$