HOMEWORK 1 SOLUTIONS

MA1132: ADVANCED CALCULUS, HILARY 2017

(1) Plug in several values of t to guess what the sketch of the parametric curve given by

$$\begin{cases} x = t^2 - t \\ y = t - 1, \end{cases}$$

where $t \in \mathbb{R}$ looks like. Now sketch the parametric curve given by the same equations, but with the restriction on t given by $-1 \leq t \leq 1$. Clearly indicate the orientation of these graphs (the direction in which t is increasing).

Solution: We simply pick some sample values and plug in for t. For example, if $t = 0$, then $x = 0$ and $y = -1$, so the point $(0, -1)$ is on the graph of the curve. These can then be plotted by hand, or using any of a variety of computer softwares. I personally prefer, and recommend, the open source program SAGE, which can be used, for example, by creating a free account at `cloud.sagemath.com`. SAGE also has great documentation, and if you are interested in learning more about it, feel free to come and see me to discuss it. In fact, even Wolfram alpha can draw many nice pictures, and although these tools shouldn't be relied on too heavily, they can be very helpful.

In this example, after sketching several points, you should notice that the graph looks something like a parabola. In fact, this isn't difficult to explain, as we can solve for t to get $t = y + 1$, and then plug into the equation for x to find

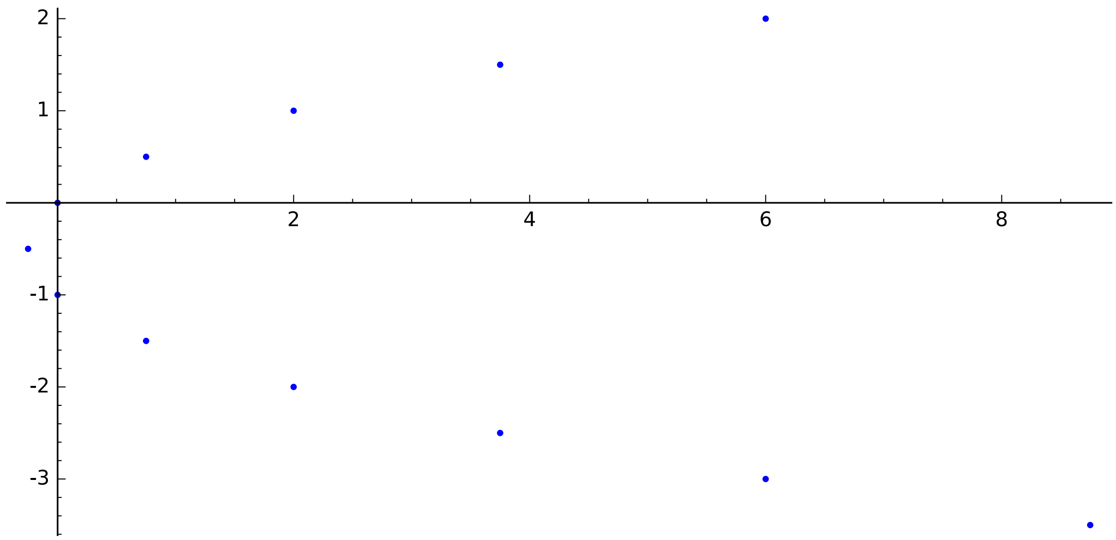
$$x = t^2 - t = (y + 1)^2 - (y + 1) = y^2 + y,$$

which is indeed the equation of a parabola "opening" to the right. To find the graph of the curve when t is restricted to lie between -1 and 1, simply plug in the points $(2, -2)$ and $(0, 0)$ corresponding to $t = -1$ and $t = 1$, respectively, and draw the segment of the parabola between these two points.

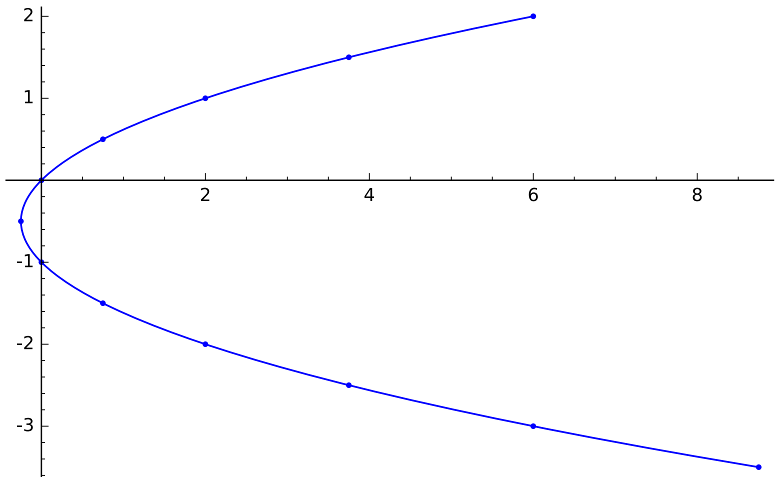
The following pictures show sample points being plugged in, as well as the corresponding parabolas with no restriction on t and when $-1 \leq t \leq 1$. The code I used in SAGE to produce these pictures is included both to indicate what is being sketched as well as for use by those interested in learning how to use the program. Note that in both cases, the orientation is that which follows the parabola "upwards" (for example, as t increases from -1 to 1, we saw that the curve goes from $(2, -2)$ to $(0, 0)$).

```
var('t')
def f(t): return (t^2-t,t-1)
point([f(-5/2),f(-2),f(-3/2),f(-1),f(-1/2),f(0),f(1/2),f(1),f(3/2),f\
   (2),f(5/2),f(3)])
```

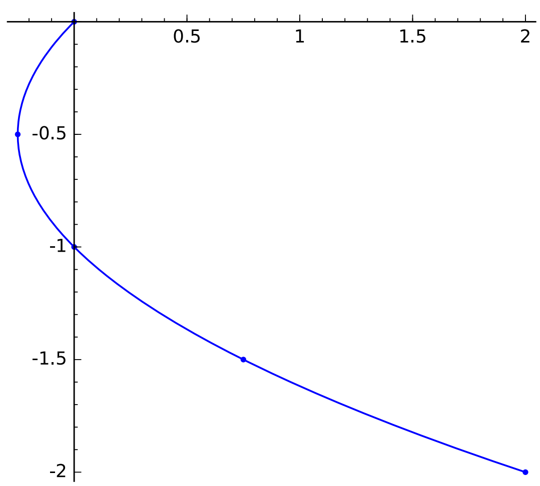
t




```
show(parametric_plot( (t^2-t,t-1), (t, -5/2,3))+point([f(-5/2),f(-2)\
   ,f(-3/2),f(-1),f(-1/2),f(0),f(1/2),f(1),f(3/2),f(2),f(5/2),f(3)])\
   )
```




```
show(parametric_plot( (t^2-t,t-1), (t, -1, 1))+point([f(-1),f(-1/2),\
    f(0),f(1/2),f(1)]))
```



(2) Prove that the sketch you drew in problem 1 matches your prediction by eliminating the parameter t (hint: solve for t in one of the parametric equations). That is, use algebra to rewrite the parametric equations in terms which allow you to immediately recognize what type of curve is given in problem 1.

Solution: The parametric equations were shown to encode the parabola $x = y^2 + y$ in the discussion of the last solution.

(3) Find the domain of

$$\vec{r}(t) = \left(\log(t^2 - 1/4), \frac{1}{t+1}, \sqrt{2t+3}\right).$$

Solution: We first find the domain of each component function. The first function, $\log(t^2 - 1/4)$, is defined when $t^2 - 1/4 > 0$, namely when $t > 1/2$ or $t < -1/2$. The second function, $\frac{1}{t+1}$, is defined for all $t \in \mathbb{R}$ except $t = -1$, where the denominator vanishes. Finally, the last component is defined whenever the argument of the square root $2t+3$, is non-negative. That is, it is defined when $t \geq -3/2$. The domain of $\vec{r}(t)$ is the intersection of these three domains, is then given by

$$\left[-\frac{3}{2}, -1\right) \cup \left(-1, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$$

(4) Find a parametric representation of the curve of intersection of the cone $x^2+y^2 = z^2$, $z > 0$, and the plane $-x + z = 1$. Indicate in a sketch what this looks like, and describe this curve.

Solution: The first picture below shows the cone $x^2 + y^2 = z^2$ with the restriction $z > 0$ (without a restriction on z, this would be a "double cone"). You have probably seen before that the intersection of a cone with a plane is a conic section, and as the pictures below illustrate, in this case the conic section will be a parabola. We can show this as follows. In the equation of the plane, solve for z to find that $z = x + 1$. Plug into the equation of the cone to find

$$x^2 + y^2 = (x + 1)^2 = x^2 + 2x + 1,$$

and thus we are looking at the parabola

$$y^2 = 2x + 1$$

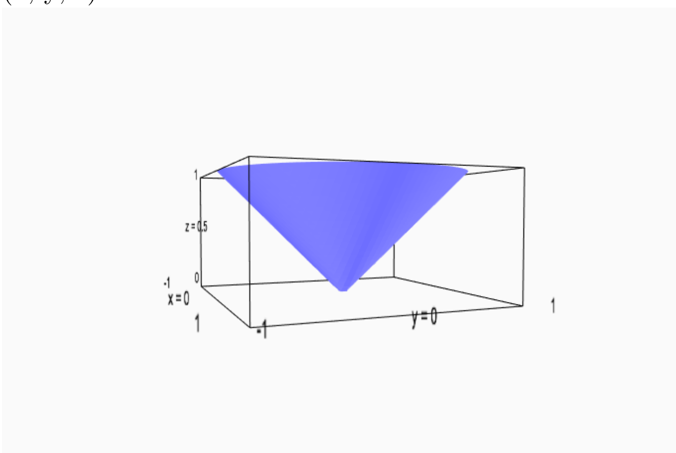
in that plane. This indicates that we should set $y = t$ (as if we set x or z equal to t, then we would have to take a square root), giving from the last equation that $x = (t^2 - 1)/2$. Plugging into the equation $z = x + 1$ then shows $z = (t^2 + 1)/2$. The parametric equation we are looking for is thus

$$\begin{cases} x = \frac{t^2-1}{2} \\ y = t \\ z = \frac{t^2+1}{2}. \end{cases}$$

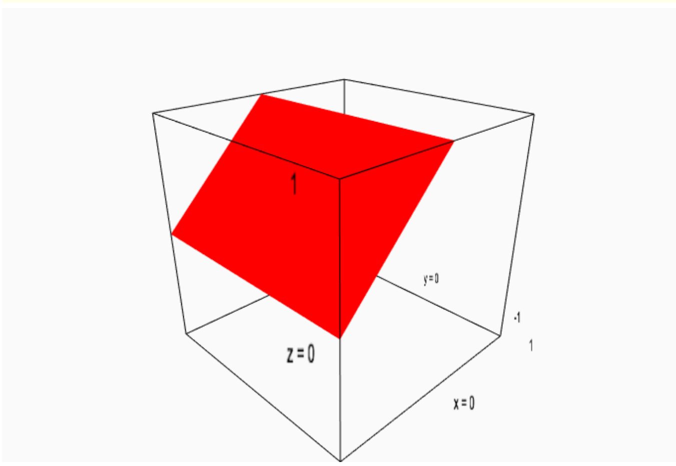
In the last picture below, I graphed this parametric curve to show that it looks like the graphed intersection of the cone and the plane. We can also double-check our work by checking that that with these equations, we indeed have $x^2+y^2 = z^2$, $z > 0$, and $-x + z = 1$.

```
var('x,y,z')
implicit_plot3d(x^2+y^2==z^2, (x,-1,1), (y,-1,1), (z,0,1))
```

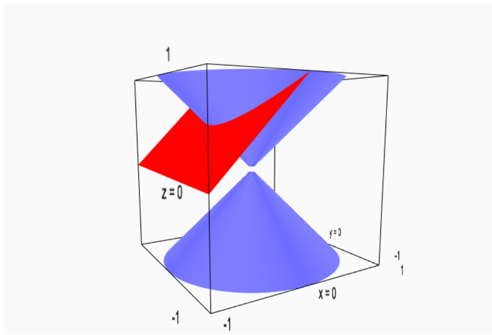
(x, y, z)




```
implicit_plot3d(-x+z==1, (x,-1,1), (y,-1,1), (z,-1,1),color='red')
```

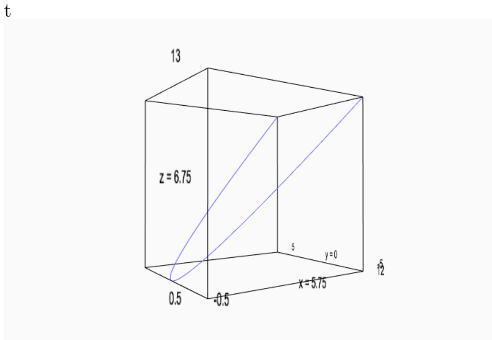



```
show(implicit_plot3d(-x+z==1, (x,-1,1), (y,-1,1), (z,0,1),color='red\
')+implicit_plot3d(x^2+y^2==z^2, (x,-1,1), (y,-1,1), (z,-1,1)))
```




```
var('t')
parametric_plot3d(((t^2-1)/2,t,(t^2+1)/2), (t,-5,5))
```

t



(5) Show that the graph of the vector-valued function

$$\vec{r}(t) = t\vec{i} + \frac{3-2t}{t}\vec{j} + \frac{(t-3)^2}{t}\vec{k}$$

$(t \neq 0)$ lies in the plane

$$x + 3y - z = 0.$$

Solution: A point on the graph of this function is given in parametric coordinates by $x = t$, $y = \frac{3-2t}{t}$, $z = (t-3)^2/t$. We need to show that any such point lies on the plane, that is, satisfies the equation $x + 3y - z = 0$. Indeed, we compute

$$x + 3y - z = t + \frac{9-6t}{t} - \frac{(t-3)^2}{t} = \frac{t^2 + 9 - 6t - (t^2 - 6t + 9)}{t} = 0.$$