HOMEWORK 10 SOLUTIONS

MA1132: ADVANCED CALCULUS, HILARY 2017

(1) Find the volume between the cone $z = r$ and the plane $z = 0$ and lying under the plane $z = 10$.

Solution: The equation $z = r$ can be written as $z = \sqrt{x^2 + y^2}$, which we have seen describes a cone. The intersection of the cone with the plane is the curve $r = z = 10$, which is a circle in the plane $z = 10$. Thus, this region under the cone is a simple x-y solid lying above the circle of radius 10 centered at the origin in the x-y plane. The interior of this circle (plus the circle itself) is the region S with $0 \leq \vartheta \leq 2\pi$ and $0 \leq r \leq 10$. For any fixed x and fixed y, the z-coordinate in the region between the cone and the plane runs from 0 to r. Thus, the volume of this region, call it R, is

$$\iiint_R dV = \iint_S \int_0^r dz dA = \int_0^{2\pi} \int_0^{10} r^2 dr d\vartheta = \frac{2000\pi}{3}.$$

(2) The *centroid* of a region in the plane is the center of mass in the case when the density function is a constant (which, since we divide out by the mass in our formula for center of mass anyways, can be assumed to be 1). Show that the centroid of a triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) is at

$$\frac{1}{3}(x_1 + x_2 + x_3, y_1 + y_2 + y_3).$$

Solution: If we rotate and translate our triangle, it won't change the geometry. Thus, we can assume that we study a triangle with vertices at $(0, 0)$, $(a, 0)$, and (b, c) for real numbers a, b, c with $a, c > 0$. The x-coordinate of the center of mass of the triangle T is given by $\iint_T x dA / \iint_T dA$. The triangle can be integrated over as as a type II region. The horizontal lines bounding this region are $y = 0$ and $y = c$, and for fixed y, x ranges from the side extending from $(0, 0)$ to (b, c), which has equation $x = \frac{by}{c}$, to the line from (b, c) to $(a, 0)$, which has equation $x = \frac{(b-a)y}{c} + a$. Thus, we find that the x-coordinate of the centroid is

$$\frac{\int_0^c \int_{\frac{by}{c}}^{\frac{(b-a)y}{c}+a} x dx dy}{\int_0^c \int_{\frac{by}{c}}^{\frac{(b-a)y}{c}+a} dx dy}$$

$$= \frac{\int_0^c \left(\frac{a^2}{2} - \frac{a(a-b)y}{c} + \frac{a(a-2b)y^2}{2c^2} \right) dy}{\int_0^c \frac{a(c-y)}{c} dy}$$

$$= \frac{\frac{(a^2+ab)c}{6}}{\frac{ac}{2}} = \frac{a}{3} + \frac{b}{3}.$$

Similarly, the y-coordinate of the centroid is

$$\frac{\int_0^c \int_{\frac{by}{c}}^{\frac{(b-a)y}{c}+a} y dx dy}{\frac{ac}{2}}$$

$$= \frac{\int_0^c \frac{a(c-y)y}{c} dy}{\frac{ac}{2}}$$

$$= \frac{\frac{ac^2}{6}}{\frac{ac}{2}} = \frac{c}{3}.$$

Thus, the centroid of this triangle lies at $\left(\frac{a+b}{3}, \frac{c}{3}\right)$. Here, the x-coordinate is the sum of the x-coordinates of the vertices divided by 3, and similarly for the y-coordinate of the centroid, as claimed.

(3) Evaluate the integral $\iiint_R xyz dV$ where R is the part of the ball $\rho \leq 1$ lying in the first octant (i.e., when $x, y, z \geq 0$).

Solution: We first recall the formula $dV = \rho^2 \sin\varphi d\rho d\varphi d\vartheta$. In spherical coordinates, the integration region is given by $(\vartheta, \varphi, \rho) \in \left[0, \frac{\pi}{2}\right] \times \left[0, \frac{\pi}{2}\right] \times [0, 1]$. Moreover, $xyz = \rho^3 \sin^2\varphi \cos\varphi \sin\vartheta \cos\vartheta$. Thus,

$$\begin{aligned}\iiint_R xyz\,dV &= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1 \rho^5\sin^3\varphi\cos\varphi\sin\vartheta\cos\vartheta\, d\rho d\varphi d\vartheta\\ &= \frac{1}{6}\cdot\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \sin^3\varphi\cos\varphi\sin\vartheta\cos\vartheta\, d\varphi d\vartheta\\ &= \frac{1}{24}\cdot\int_0^{\frac{\pi}{2}} \sin\vartheta\cos\vartheta\left[\sin^4\varphi\right]_{\varphi=0}^{\frac{\pi}{2}} d\vartheta\\ &= \frac{1}{24}\cdot\int_0^{\frac{\pi}{2}}\sin\vartheta\cos\vartheta\, d\vartheta\\ &= -\frac{1}{48}[\cos^2\vartheta]_{\vartheta=0}^{\frac{\pi}{2}} = \frac{1}{48}.\end{aligned}$$

(4) Change variables to compute

$$\iint_R xy\,dA,$$

where R is the parallelogram with vertices at $(0,\pm1)$, $(\pm2,0)$ by turning the integration region into a rectangle with sides parallel to the u and v axes for some coordinates u,v.

Solution: We first write down equations for the lines forming the sides of the parallelogram. These are $x+2y=-2$ (lower-left side), $x+2y=2$ (upper-right), $x-2y=-2$ (upper-left), and $x-2y=2$ (lower-right). We thus are motivated to set $u=x+2y$, $v=x-2y$. In this coordinate system, the sides of the parallelogram are the lines $u,v=\pm2$. Solving for x and y gives $x=\frac{u+v}{2}$, $y=\frac{u-v}{4}$. Thus, the Jacobian is

$$J(u,v)=\det\begin{pmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{pmatrix} = \det\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{4} & -\frac{1}{4}\end{pmatrix} = -\frac{1}{4}.$$

Thus, the region R is just the image of the rectangle $[-2,2]\times[-2,2]$ in the u-v plane, and so our integral becomes

$$\frac{1}{8}\int_{-2}^{2}\int_{-2}^{2}(u^2-v^2)\frac{dudv}{4} = \frac{1}{32}\int_{-2}^{2}\left[\frac{u^3}{3}-uv^2\right]_{u=-2}^{2} dv = \int_{-2}^{2}\left(-\frac{v^2}{8}+\frac{1}{6}\right)dv = 0.$$

(5) Find the area of an elliptical region given by $\frac{x^2}{a^2}+\frac{y^2}{b^2}\leq 1$ by finding a suitable change of variables which transforms the problem into a problem of integrating over a circular region.

Solution:

If we set $x=au$, $y=bv$, then the boundary ellipse becomes $1=\frac{x^2}{a^2}+\frac{y^2}{b^2}=u^2+v^2$. Thus, the elliptical region is the image of the circle $u^2+v^2\leq 1$ under this transformation. The Jacobian is

$$J(u,v) = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \det \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = ab.$$

Thus, the area of the ellipse is $\int_0^{2\pi} \int_0^1 r \cdot ab \cdot dr d\vartheta = \pi ab$.