Linear algebra I
2012 final exam solutions

1a. The columns of A are linearly dependent if and only if

$$\det \begin{bmatrix} 1 & x & 1 \\ 1 & 1 & x \\ 1 & 3 & 1 \end{bmatrix} = 0 \quad \iff \quad x^2 - 4x + 3 = 0 \quad \iff \quad x = 1, 3.$$

1b. When $x = 2$, we can find the inverse of the given matrix using the row reduction

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 1 & 3 & 1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & -1 & -3 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{array}\right].$$

2a. Suppose that A is $n \times n$ lower triangular. Using row reduction, one finds that

$$\det A = a_{11} \det \begin{bmatrix} 1 & & & \\ a_{21} & a_{22} & & \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = a_{11} \det \begin{bmatrix} 1 & & & \\ 0 & a_{22} & & \\ \vdots & \vdots & \ddots & \\ 0 & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

We now proceed in the same manner, factoring out a_{22} and clearing all entries below the pivot. Applying this approach repeatedly, we end up with $\det A = a_{11}a_{22} \cdots a_{nn}$.

2b. Using the identity $A \cdot \operatorname{adj} A = (\det A) I_n$, we get

$$\det(A \cdot \operatorname{adj} A) = (\det A)^n \quad \Longrightarrow \quad (\det A) \cdot \det(\operatorname{adj} A) = (\det A)^n.$$

Since A is invertible, however, $\det A \neq 0$ and this implies $\det(\operatorname{adj} A) = (\det A)^{n-1}$.

3a. The reduced row echelon form of A is

$$\begin{bmatrix} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus, the null space and the column space of A are

$$\mathcal{N}(A) = \operatorname{Span} \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}, \qquad \mathcal{C}(A) = \operatorname{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

3b. Assuming that y is in the column space of B, we have

$$\begin{aligned} y = Bx \text{ for some } x \in \mathbb{R}^n \quad &\implies \quad Ay = A(Bx) \text{ for some } x \in \mathbb{R}^n \\ &\implies \quad Ay = (AB)x = 0 \text{ for some } x \in \mathbb{R}^n \\ &\implies \quad y \text{ is in the null space of } A. \end{aligned}$$

This shows that the column space of B is contained in the null space of A, so

$$\operatorname{rk}(B) \leq \dim \mathcal{N}(A) \quad \implies \quad \operatorname{rk}(A) + \operatorname{rk}(B) \leq n.$$

4a. With respect to the standard basis, we have

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

With respect to the basis B, we have

$$T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \qquad T\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

and we need to express those in terms of the basis B. Using the row reduction

$$\begin{bmatrix} 1 & 1 & -2 & -3 \\ 2 & 3 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -7 & -10 \\ 0 & 1 & 5 & 7 \end{bmatrix},$$

we conclude that the desired matrix is $A = \begin{bmatrix} -7 & -10 \\ 5 & 7 \end{bmatrix}$.

4b. Letting A be the matrix whose ith column is $T(\boldsymbol{e}_i)$ for each i, we get

$$T(\boldsymbol{x}) = T\left(\sum_{i=1}^{n} x_i \boldsymbol{e}_i\right) = \sum_{i=1}^{n} x_i T(\boldsymbol{e}_i) = \sum_{i=1}^{n} x_i A\boldsymbol{e}_i = A\boldsymbol{x}.$$