Linear algebra I
2011 final exam solutions

1. When it comes to the inverse of the given matrix, we use row reduction to get

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 3 & 9 & 0 & 1 & 0 \\ 1 & 4 & 16 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -2 & 1 \\ 0 & 1 & 0 & -7/6 & 5/2 & -4/3 \\ 0 & 0 & 1 & 1/6 & -1/2 & 1/3 \end{array}\right].$$

Let $f(t) = a + bt + ct^2$ be the desired polynomial. Then we must have

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 11 \end{bmatrix},$$

where the leftmost matrix is the given one. Multiplying by its inverse, we find

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 \\ -7/6 & 5/2 & -4/3 \\ 1/6 & -1/2 & 1/3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 11 \end{bmatrix} = \begin{bmatrix} 13 \\ -95/6 \\ 23/6 \end{bmatrix}.$$

2. A permutation is even, if the associated permutation matrix has determinant 1. The first permutation is (147)(23)(56) and it is even, the second one is (176)(28)(345) and it is odd, while the last permutation is (14567328) and it is also odd.

3a. Using the definition of matrix multiplication, we get

$$\operatorname{tr}(AB) = \sum_{k=1}^{n} (AB)_{kk} = \sum_{k=1}^{n} \sum_{m=1}^{n} A_{km} B_{mk}$$

and then a similar computation gives

$$\operatorname{tr}(BA) = \sum_{m=1}^{n} (BA)_{mm} = \sum_{m=1}^{n} \sum_{k=1}^{n} B_{mk} A_{km} = \operatorname{tr}(AB).$$

3b. Repeated applications of part (a) give

$$\operatorname{tr}(ABC) = \operatorname{tr}(CAB) = \operatorname{tr}(BCA), \qquad \operatorname{tr}(ACB) = \operatorname{tr}(BAC) = \operatorname{tr}(CBA)$$

so there can be at most two distinct traces. A simple example for which all traces are equal is the case $A = 0$, in which case all traces are zero. A simple example for which two distinct traces exist is the case

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

In that case, $AB = B$, $AC = 0$ and $BC = A$, so one easily finds that

$$\operatorname{tr}(ABC) = \operatorname{tr} A = 1, \qquad \operatorname{tr}(ACB) = \operatorname{tr} 0 = 0.$$

4. The reduced row echelon form of the given matrix is

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & 0 & 2 & 5 & 5 \\ 0 & 0 & 1 & 0 & 2 & 1 & -1 \\ 0 & 0 & 0 & 1 & -5 & -8 & -5 \end{bmatrix},$$

so its null space is described by the equations

$$\begin{aligned} x_1 &= -\ x_5 - 2x_6 - 3x_7, \\ x_2 &= -2x_5 - 5x_6 - 5x_7, \\ x_3 &= -2x_5 - \ x_6 + \ x_7, \\ x_4 &= \ 5x_5 + 8x_6 + 5x_7. \end{aligned}$$

This means that the null space is three-dimensional and that a basis is

$$\boldsymbol{v}_1 = \begin{bmatrix} -1 \\ -2 \\ -2 \\ 5 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} -2 \\ -5 \\ -1 \\ 8 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{v}_3 = \begin{bmatrix} -3 \\ -5 \\ 1 \\ 5 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$