Linear algebra I
2010 final exam solutions

1. We can compute the inverse of A using the row reduction

$$\left[\begin{array}{rrr:rrr} 1 & -1 & 1 & 1 & 0 & 0 \\ -1 & 2 & -2 & 0 & 1 & 0 \\ 1 & -2 & 3 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{rrr:rrr} 1 & 0 & 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

and then we can solve the system $Ax = b$ as

$$x = A^{-1}b = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 6 \end{bmatrix}.$$

2. The first permutation is $(1, 5, 9, 2)(3, 10, 7, 11)$ and it is even; the second permutation is $(1, 10, 2, 3, 9, 11, 5, 7)$ and it is odd.

3a. A set S of vectors is called complete, if every element of V can be written as a linear combination of the vectors in S. It is called linearly independent, if no vector in S is a linear combination of the other vectors in S. It is called a basis, if it is both complete and linearly independent.

- Suppose now that $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_k\}$ is complete and let $S' = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_k, \boldsymbol{v}\}$. To show that S' is complete, let $\boldsymbol{w} \in V$ be an arbitrary vector. Then we can write

$$\boldsymbol{w} = x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + \ldots + x_k\boldsymbol{v}_k$$

for some scalar coefficients x_i by completeness. Writing this equation as

$$\boldsymbol{w} = x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + \ldots + x_k\boldsymbol{v}_k + 0\boldsymbol{v},$$

we see that $\boldsymbol{w}$ is a linear combination of vectors in S'. Thus, S' is also complete.

3b. To prove completeness, suppose $\boldsymbol{z} \in V$. We have to find scalars x_1, x_2, x_3 such that

$$\boldsymbol{z} = x_1(\boldsymbol{u} + \boldsymbol{v}) + x_2(\boldsymbol{u} - \boldsymbol{w}) + x_3(2\boldsymbol{v} + \boldsymbol{w}).$$

Write the last equation in the form

$$\boldsymbol{z} = (x_1 + x_2)\boldsymbol{u} + (x_1 + 2x_3)\boldsymbol{v} + (x_3 - x_2)\boldsymbol{w}.$$

Since the set $S = \{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ is complete, there do exist scalars y_1, y_2, y_3 with

$$\boldsymbol{z} = y_1\boldsymbol{u} + y_2\boldsymbol{v} + y_3\boldsymbol{w}.$$

Thus, it suffices to show that there exist scalars x_1, x_2, x_3 with

$$x_1 + x_2 = y_1, \qquad x_1 + 2x_3 = y_2, \qquad x_3 - x_2 = y_3.$$

Note that this is a linear system of the form $Ax = y$, where

$$\det A = \det \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & -1 & 1 \end{bmatrix} = 2 - 1 = 1.$$

In particular, A is invertible and the system has a unique solution, as needed.

- Since V is spanned by 3 vectors, its dimension is at most 3, namely $\dim V = 0, 1, 2, 3$.

4. We are given $A(\boldsymbol{v}_1), A(\boldsymbol{v}_2), A(\boldsymbol{v}_3)$ and we need to compute $A(\boldsymbol{v})$. Thus, we shall need to express $\boldsymbol{v}$ as a linear combination of the $\boldsymbol{v}_i$'s. Using the row reduction

$$\begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 & \boldsymbol{v} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & -1 & 2 & 0 \\ 1 & 0 & 2 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix},$$

we find that $\boldsymbol{v} = -5\boldsymbol{v}_1 + 3\boldsymbol{v}_2 + 4\boldsymbol{v}_3$. Since A is linear, this also implies that

$$A(\boldsymbol{v}) = -5A(\boldsymbol{v}_1) + 3A(\boldsymbol{v}_2) + 4A(\boldsymbol{v}_3),$$

namely that

$$A(\boldsymbol{v}) = -5 \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 2 \\ 3 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 20 \\ 7 \\ -9 \end{bmatrix}.$$