3 Unital abelian Banach algebras

3.1 Characters and maximal ideals

Let A be a unital abelian Banach algebra.

3.1.1 Definition. A *character* on A is a non-zero homomorphism $A \to \mathbb{C}$; that is, a non-zero linear map $\tau\colon A \to \mathbb{C}$ which satisfies $\tau(ab) = \tau(a)\tau(b)$ for $a, b \in A$. We write $\Omega(A)$ for the set of characters on A.

3.1.2 Example. Let $A = C(X)$ where X is a compact topological space. For each $x \in X$, the map $\varepsilon_x\colon A \to \mathbb{C}$, $f \mapsto f(x)$ is a character on A.

3.1.3 Remark. This definition makes sense even if A is not abelian. However, $\Omega(A)$ is often not very interesting in that case.

For example, if $A = M_n(\mathbb{C})$ and $n > 1$ then $\Omega(A) = \emptyset$. Indeed, it is not hard to show that A is spanned by $\{ab - ba\colon a, b \in A\}$. If $\varphi\colon A \to \mathbb{C}$ is a homomorphism, then $\varphi(ab - ba) = \varphi(a)\varphi(b) - \varphi(b)\varphi(a) = 0$, so $\varphi = 0$ by linearity; hence $\Omega(A) = \emptyset$.

3.1.4 Lemma. *If $\tau \in \Omega(A)$ then τ is continuous. More precisely,*

$$\|\tau\| = \tau(1) = 1.$$

In particular, $\Omega(A)$ is a subset of the closed unit ball of A^*.

Proof. Observe that $\tau(1) = \tau(1^2) = \tau(1)^2$, so $\tau(1) \in \{0, 1\}$. If $\tau(1) = 0$ then $\tau(a) = \tau(a1) = \tau(a)\tau(1) = 0$ for any $a \in A$, so $\tau = 0$. But $\tau \in \Omega(A)$ so $\tau \neq 0$, which is a contradiction. So $\tau(1) = 1$.

We have $\tau(\operatorname{Inv} A) \subseteq \operatorname{Inv} \mathbb{C} = \mathbb{C} \setminus \{0\}$ by 1.5.12(i). For any $a \in A$ we have $\tau(\tau(a)1 - a) = \tau(a)\tau(1) - \tau(a) = 0$, so $\tau(a)1 - a \notin \operatorname{Inv} A$. Hence $\tau(a) \in \sigma(a)$, so $|\tau(a)| \leq \|a\|$ by Theorem 1.3.5 and so $\|\tau\| \leq 1$. Since $|\tau(1)| = 1$ we conclude that $\|\tau\| = 1$.

Any $\tau \in \Omega(A)$ is a linear map $A \to \mathbb{C}$ by definition, and we have shown that it continuous with norm 1. Hence $\Omega(A)$ is contained in the closed unit ball of A^*. $\square$

3.1.5 Definition. The *Gelfand topology* on $\Omega(A)$ is the subspace topology obtained from the weak* topology on A^*.

We will always equip $\Omega(A)$ with this topology.

3.1.6 Theorem. *$\Omega(A)$ is a compact Hausdorff space.*

Proof. We observed in Remark 2.4.2(iii) that the weak* topology is Hausdorff, so $\Omega(A)$ is a Hausdorff space. By Lemma 3.1.4, $\Omega(A)$ is contained in the unit ball of A^*, which is compact in the weak* topology by Theorem 2.4.3. A closed subset of a compact set is compact, so it suffices to show that $\Omega(A)$ is weak* closed in A^*. But

$$\begin{aligned}\Omega(A) &= \{\tau \in A^* \colon \tau(1) = 1,\ \tau(ab) = \tau(a)\tau(b) \text{ for } a, b \in A\} \\ &= \{\tau \in A^* \colon \tau(1) = 1\} \cap \bigcap_{a,b\in A} \{\tau \in A^* \colon \tau(ab) - \tau(a)\tau(b) = 0\} \\ &= J_1^{-1}(1) \cap \bigcap_{a,b\in A} (J_{ab} - J_a \cdot J_b)^{-1}(0).\end{aligned}$$

Each evaluation functional $J_a \colon A^* \to \mathbb{C}$, $\tau \mapsto \tau(a)$ is weak* continuous, so the maps $J_{ab} - J_a \cdot J_b$ are also weak* continuous. Hence the sets in this intersection are all weak* closed, so $\Omega(A)$ is weak* closed. $\square$

3.1.7 Lemma. *Let A be a unital abelian Banach algebra.*

(i). If $\tau \in \Omega(A)$ then $\ker \tau$ is a maximal ideal of A.

(ii). If M is a maximal ideal of A, then the map $\mathbb{C} \to A/M$, $\lambda \mapsto \lambda 1 + M$ is an isometric isomorphism.

Proof. (i) Let $\tau \in \Omega(A)$. Since τ is a non-zero homomorphism, its kernel $I = \ker \tau$ is a proper ideal of A. Suppose that J is an ideal of A with $I \subsetneq J$ and let $a \in J \setminus I$. Then $\tau(a) \neq 0$, so $b = \tau(a)^{-1}a \in J$ and $\tau(b) = 1$. Since $\tau(1) = 1$ by Lemma 3.1.4, we have $1 - b \in I$, so $1 = b + 1 - b \in J$. By Lemma 1.5.5, $J = A$. This shows that I is not contained in any strictly larger proper ideal of A, so I is a maximal ideal of A.

(ii) Let M be a maximal ideal of A. By Theorems 1.5.6(ii) and 1.5.3, A/M is unital Banach algebra with unit $1 + M$. If $a + M$ is a non-zero element of A/M then $a \in A \setminus M$. Let $I = \{ab + m \colon m \in M,\ b \in A\}$. Since A is abelian and M is an ideal, it is easy to see that I is an ideal of A, and $M \subsetneq I$. Since M is a maximal ideal, $I = A$. So $1 \in I$, and $ab + m = 1$ for some $b \in A$ and $m \in M$. Now

$$(a + M)(b + M) = ab + M = ab + m + M = 1 + M,$$

which is the unit of A/M. Hence $b + M = (a + M)^{-1}$ and $a + M$ is invertible in A/M. By the Gelfand-Mazur theorem 1.3.6, $A/M = \mathbb{C}1_{A/M} = \mathbb{C}(1 + M)$.

It is very easy to check that the map $\mathbb{C} \to A/M$, $\lambda \mapsto \lambda 1 + M$ is an isometric homomorphism, and we have just shown that it is surjective. Hence it is an isometric isomorphism. $\square$

3.1.8 Theorem. *Let A be a unital abelian Banach algebra. The mapping*

$$\tau \mapsto \ker \tau$$

is a bijection from $\Omega(A)$ onto the set of maximal ideals of A.

Proof. If $\tau \in \Omega(A)$ then $\ker \tau$ is a maximal ideal of A, by Lemma 3.1.7(i). Hence the mapping is well-defined.

The mapping $\tau \mapsto \ker \tau$ is injective, since if τ_1 and τ_2 are in $\Omega(A)$ with $\ker \tau_1 = \ker \tau_2$, then for any $a \in A$ we have $a - \tau_2(a)1 \in \ker \tau_2 = \ker \tau_1$ so $\tau_1(a - \tau_2(a)1) = 0$, hence $\tau_1(a) = \tau_2(a)$; so $\tau_1 = \tau_2$.

We now show that the mapping is surjective. Let M be a maximal ideal of A, and let $q\colon A \to A/M$, $a \mapsto a + M$ be the corresponding quotient map. Observe that q is a homomorphism and $\ker q = M$. By Lemma 3.1.7(ii), the map $\theta\colon \mathbb{C} \to A/M$, $\lambda \mapsto \lambda 1 + M$ is an isomorphism. Let $\tau = \theta^{-1} \circ q\colon A \to \mathbb{C}$. Since τ is the composition of two homomorphisms, it is a homomorphism, and $\tau(1) = \theta^{-1}(q(1)) = \theta^{-1}(1 + M) = 1$, so $\tau \neq 0$. Hence $\tau \in \Omega(A)$. Since θ is an isomorphism, we have $\ker \tau = \ker q = M$.

This shows that $\tau \mapsto \ker \tau$ is a bijection from $\Omega(A)$ onto the set of all maximal ideals of A. □

3.1.9 Examples. (i). Let X be a compact Hausdorff space. For $x \in X$, the map $\varepsilon_x\colon C(X) \to \mathbb{C}$, $f \mapsto f(x)$ is a nonzero homomorphism, so $\{\varepsilon_x\colon x \in X\} \subseteq \Omega(C(X))$. We claim that we have equality.

For $x \in X$, let $M_x = \ker \varepsilon_x = \{f \in C(X)\colon f(x) = 0\}$, which is a maximal ideal of $C(X)$ by Theorem 3.1.8. Let I be an ideal of $C(X)$. If $I \not\subseteq M_x$ for every $x \in X$, then for each $x \in X$, there is $f_x \in I$ with $f_x \neq 0$. Since I is an ideal, $g_x = |f_x|^2 = \overline{f_x} f_x \in I$, and since g_x is continuous and non-negative with $g_x(x) > 0$, there is an open set U_x with $x \in U_x$ and $g_x(y) > 0$ for all $y \in U_x$. As x varies over X, the open sets U_x cover X. Since X is compact, there is $n \geq 1$ and $x_1, \dots, x_n \in X$ such that $U_{x_1}, \dots, U_{x_n}$ cover X. Let $g = g_{x_1} + \cdots + g_{x_n}$. Then $g \in I$ and $g(x) > 0$ for all $x \in X$, so g is invertible in $C(X)$. Hence $I = C(X)$.

This shows that every proper ideal I of $C(X)$ is contained in M_x for some $x \in X$. Let $\tau \in \Omega(C(X))$. Since $\ker \tau$ is a maximal (proper) ideal, we must have $\ker \tau = M_x$ for some $x \in X$, so $\tau = \varepsilon_x$ by Theorem 3.1.8.

Consider the map $\theta\colon X \to \Omega(C(X))$, $x \mapsto \varepsilon_x$. We have just shown that this is surjective. Since X is compact and Hausdorff, $C(X)$ separates the points of X by Urysohn's lemma. Hence if $\varepsilon_x = \varepsilon_y$ then $f(x) = f(y)$ for all $f \in C(X)$, so $x = y$. Hence θ is a bijection.

We claim that θ is a homeomorphism. Indeed, θ is continuous since for $f \in C(X)$ and $x \in X$ we have

$$J_f(\theta(x)) = J_f(\varepsilon_x) = \varepsilon_x(f) = f(x),$$

so $J_f \circ \theta$ (or, more precisely, $J_f|_{\Omega(C(X))} \circ \theta$) is continuous $X \to \mathbb{C}$ for every $f \in C(X)$. By Proposition 2.2.6, θ is continuous. Since X is compact and $\Omega(C(X))$ is Hausdorff, Lemma 2.1.1 shows that θ is a homeomorphism.

(ii). Recall that $A(\overline{\mathbb{D}})$ denotes the disc algebra. If $w \in \mathbb{D}$ then $\varepsilon_w : A(\overline{\mathbb{D}}) \to \mathbb{C}$, $f \mapsto f(w)$ is a character on $A(\overline{\mathbb{D}})$.

Again, we claim that every character arises in this way. To see this, consider the function $z \in A(\overline{\mathbb{D}})$ defined by $z(w) = w$, $w \in \overline{\mathbb{D}}$. If $\tau \in \Omega(A(\overline{\mathbb{D}}))$ then $\tau(1) = 1$ and $|\tau(z)| \le \|z\| = 1$, so $\tau(z) \in \overline{\mathbb{D}}$. It is not hard to show that the polynomials $\mathbb{D} \to \mathbb{C}$ form a dense unital subalgebra of $A(\overline{\mathbb{D}})$. If $p\colon \mathbb{D} \to \mathbb{C}$ is a polynomial then $p = \lambda_0 1 + \lambda_1 z + \cdots + \lambda_n z^n$ for constants λ_i, so $\tau(p) = \lambda_0 + \lambda_1\tau(z) + \cdots + \lambda_n\tau(z)^n = p(\tau(z))$. Since the polynomials are dense in $A(\overline{\mathbb{D}})$, this shows that $\tau(f) = f(\tau(z))$ for all $f \in A(\overline{\mathbb{D}})$, so $\tau = \varepsilon_{\tau(z)}$. Hence $\Omega(A(\overline{\mathbb{D}})) = \{\varepsilon_w \colon w \in \overline{\mathbb{D}}\}$. Just as in (i), it is easy to see that the map $w \mapsto \varepsilon_w$ is a homeomorphism $\overline{\mathbb{D}} \to \Omega(A(\overline{\mathbb{D}}))$.

(iii). Recall the abelian Banach algebra $\ell^1(\mathbb{Z})$ with product $*$ from Example 1.1.3(v). For $n \in \mathbb{Z}$, let $e_n = (\delta_{mn})_{m\in\mathbb{Z}}$. Then $e_n \in \ell^1(\mathbb{Z})$, and the linear span of $\{e_n \colon n \in \mathbb{Z}\}$ is dense in $\ell^1(\mathbb{Z})$. Moreover, it is easy to check that $e_n * e_m = e_{n+m}$ for $n, m \in \mathbb{Z}$. In particular, $e_0 * e_m = e_m$, hence e_0 is the unit for $\ell^1(\mathbb{Z})$.

If $\tau \in \Omega(\ell^1(\mathbb{Z}))$ then $\tau(e_0) = 1$. Moreover, $e_n * e_{-n} = e_0$ so $e_n = (e_{-n})^{-1}$ and $|\tau(e_n)| \le \|e_n\| = 1$ for each $n \in \mathbb{Z}$, so

$$1 \le |\tau(e_{-n})|^{-1} = |\tau((e_{-n})^{-1})| = |\tau(e_n)| \le 1.$$

So we have equality. In particular, $|\tau(e_1)| = 1$, so $\tau(e_1) \in \mathbb{T}$. Since $e_n = e_1^n$, we have $\tau(e_n) = \tau(e_1)^n$. Hence if $x \in \ell^1(\mathbb{Z})$ then

$$\tau(x) = \tau\left(\sum_{n\in\mathbb{Z}} x_n e_n\right) = \sum_{n\in\mathbb{Z}} x_n \tau(e_1)^n.$$

So τ is determined by the complex number $\tau(e_1) \in \mathbb{T}$. Conversely, given any $z \in \mathbb{T}$, there is a character $\tau_z \in \Omega(\ell^1(\mathbb{Z}))$ with $\tau_z(e_1) = z$. Indeed, let $A_0 = \operatorname{span}\{e_n \colon n \in \mathbb{Z}\}$, which is a dense subalgebra of $\ell^1(\mathbb{Z})$. Let

$\tau_0\colon A_0 \to \mathbb{C}$ be the unique linear map such that $\tau_0(e_n) = z^n$ for all $n \in \mathbb{Z}$. If $x, y \in A_0$, then

$$\tau_0(x * y) = \tau_0\Big(\sum_{m,n\in\mathbb{Z}} x_m y_{n-m} e_n\Big) = \sum_{m,n\in\mathbb{Z}} x_m y_{n-m} z^n$$
$$= \sum_{m,n\in\mathbb{Z}} x_m z^m \, y_{n-m} z^{n-m} = \tau_0(x)\tau_0(y),$$

so τ_0 is a homomorphism, and

$$|\tau_0(x)| = \Big|\sum_{n\in\mathbb{Z}} x_n z^n\Big| \le \sum_{n\in\mathbb{Z}} |x_n| = \|x\|,$$

so τ_0 is continuous. Hence τ_0 extends to a continuous linear homomorphism $\tau_z\colon \ell^1(\mathbb{Z}) \to \mathbb{C}$, which is a character on $\ell^1(\mathbb{Z})$. Clearly, $\tau_z(e_1) = z$. This shows that the map $\theta\colon \mathbb{T} \to \Omega(\ell^1(\mathbb{Z}))$, $z \mapsto \tau_z$ is a bijection. We claim that θ is a homeomorphism. Indeed, θ is continuous since for $x \in \ell^1(\mathbb{Z})$ and $z \in \mathbb{T}$ we have

$$J_x(\theta(z)) = J_x(\tau_z) = \tau_z(x) = \sum_{n\in\mathbb{Z}} x_n z^n.$$

Since $x \in \ell^1(\mathbb{Z})$, this series converges absolutely (for $z \in \mathbb{T}$). The partial sums of the series are continuous functions $\mathbb{T} \to \mathbb{C}$, so $z \mapsto J_x(\theta(z))$ is continuous $\mathbb{T} \to \mathbb{C}$. By Proposition 2.2.6, θ is continuous. Since $\mathbb{T}$ is compact and $\Omega(\ell^1(\mathbb{Z}))$ is Hausdorff, Lemma 2.1.1 shows that θ is a homeomorphism.

The next lemma is purely algebraic.

3.1.10 Lemma. *Let A be a unital abelian Banach algebra and let $a \in A$. The following are equivalent:*

(i). $a \notin \operatorname{Inv} A$;

(ii). $a \in I$ for some proper ideal I of A;

(iii). $a \in M$ for some maximal ideal M of A.

Proof. (i) $\implies$ (ii): If $a \notin \operatorname{Inv} A$, consider the set $I = \{ab\colon b \in A\}$. Since A is abelian, this is an ideal of A, and since A is unital we have $a = a1 \in I$. If $1 \in I$ then $ab = 1$ for some $b \in A$, so $a \in \operatorname{Inv} A$, a contradiction. So $1 \notin I$ and I is a proper ideal.

(ii) $\implies$ (iii): Suppose that $a \in I$ where I is a proper ideal of A. By Remark 1.5.7(ii), $I \subseteq M$ for some maximal ideal M of A; so $a \in M$.

(iii) $\implies$ (i): If M is a maximal ideal of A then M is a proper ideal, so $M \cap \operatorname{Inv} A = \emptyset$ by Lemma 1.5.5. Hence $a \notin \operatorname{Inv} A$ for every $a \in M$. □

3.1.11 Corollary. *Let A be a unital abelian Banach algebra and let $a \in A$.*

(i). $a \in \operatorname{Inv} A$ *if and only if* $\tau(a) \neq 0$ *for all* $\tau \in \Omega(A)$.

(ii). $\sigma(a) = \{\tau(a) \colon \tau \in \Omega(A)\}$.

(iii). $r(a) = \sup_{\tau \in \Omega(A)} |\tau(a)|$.

Proof. (i) We have:

$$\begin{aligned} a \in \operatorname{Inv} A &\iff a \notin M \text{ for all maximal ideals } M \text{ of } A, \text{ by Lemma 3.1.10} \\ &\iff a \notin \ker \tau \text{ for all } \tau \in \Omega(A), \text{ by Theorem 3.1.8} \\ &\iff \tau(a) \neq 0 \text{ for all } \tau \in \Omega(A). \end{aligned}$$

(ii) This follows from the equivalences:

$$\begin{aligned} \lambda \in \sigma(a) &\iff \lambda - a \notin \operatorname{Inv} A \\ &\iff \tau(\lambda - a) = 0 \text{ for some } \tau \in \Omega(A), \text{ by (i)} \\ &\iff \lambda = \tau(a) \text{ for some } \tau \in \Omega(A), \text{ since } \tau(\lambda) = \lambda \text{ by Lemma 3.1.4.} \end{aligned}$$

(iii) follows immediately from (ii) and the definition of $r(a)$. □

3.2 The Gelfand representation

3.2.1 Definition. Let A be a unital abelian Banach algebra. For $a \in A$, the *Gelfand transform of* a is the mapping

$$\widehat{a} \colon \Omega(A) \to \mathbb{C}, \quad \tau \mapsto \tau(a).$$

In other words, $\widehat{a} = J_a|_{\Omega(A)}$.

3.2.2 Examples. (i). Let X be a compact Hausdorff space. We have seen that the map $X \to \Omega(C(X))$, $x \mapsto \varepsilon_x$ is a homeomorphism. If $f \in C(X)$ then

$$\widehat{f} \colon \Omega(C(X)) \to \mathbb{C}, \quad \varepsilon_x \mapsto \varepsilon_x(f) = f(x).$$

This means that, if we identify $\Omega(C(X))$ with X by pretending that $x = \varepsilon_x$, then $\widehat{f} = f$.

(ii). We have seen that $\Omega(\ell^1(\mathbb{Z}))$ can be identified with $\mathbb{T}$, by pretending that $z \in \mathbb{T}$ is the same as the character $\tau_z \colon \ell^1(\mathbb{Z}) \to \mathbb{C}$, $x \mapsto \sum_{n \in \mathbb{Z}} x_n z^n$. Hence for $x \in \ell^1(\mathbb{Z})$, we have

$$\widehat{x} \colon \Omega(\ell^1(\mathbb{Z})) \to \mathbb{C}, \quad \tau_z \mapsto \tau_z(x) = \sum_{n \in \mathbb{Z}} x_n z^n.$$

If we write $z = e^{i\theta}$ and $x_n = x(n)$ then this takes the form

$$\widehat{x}(e^{i\theta}) = \sum_{n\in\mathbb{Z}} x(n)e^{in\theta},$$

so $\widehat{x}$ maybe viewed as the inverse Fourier transform of x.

3.2.3 Theorem. *Let A be a unital abelian Banach algebra. For each $a \in A$, the Gelfand transform $\widehat{a}$ is in $C(\Omega(A))$. Moreover, the mapping*

$$\gamma\colon A \to C(\Omega(A)), \quad a \mapsto \widehat{a}$$

is a unital, norm-decreasing (and hence continuous) homomorphism, and for each $a \in A$ we have

$$\sigma_A(a) = \sigma_{C(\Omega(A))}(\widehat{a}) = \{\widehat{a}(\tau)\colon \tau \in \Omega(A)\} \quad \textit{and} \quad r(a) = \|\widehat{a}\|.$$

Proof. By the definition of the topology on $\Omega(A)$, each $\widehat{a}$ is in $C(\Omega(A))$. It is easy to see that γ is a homomorphism, and it is unital by Lemma 3.1.4. The identification of $\sigma_A(a)$ with $\sigma_{C(\Omega(A))}(\widehat{a})$ follows from Corollary 3.1.11(ii) and Example 1.3.2(ii). Now $r(a) = \|\widehat{a}\| \leq \|a\|$ by Corollary 3.1.11(iii) and Remark 1.3.10, so γ is linear and norm-decreasing, hence continuous. $\square$

3.2.4 Definition. If A is a unital abelian Banach algebra then the unital homomorphism $\gamma\colon A \to C(\Omega(A))$, $a \mapsto \widehat{a}$ is called the *Gelfand representation* of A.

3.2.5 Remark. In general, the Gelfand representation is neither injective nor surjective. For example, the Gelfand representation of the disc algebra is the inclusion $A(\overline{\mathbb{D}}) \to C(\overline{\mathbb{D}})$ which is not surjective.