2 A topological interlude

2.1 Topological spaces

Recall that a topological space is a set X with a topology: a collection $\mathcal{T}$ of subsets of X, known as *open sets*, such that $\emptyset$ and X are open, and finite intersections and arbitrary unions of open sets are open. We call a set $F \subseteq X$ *closed* if its complement $X \setminus F$ is open.

An open cover C of X is a collection of open subsets of X whose union is X. A finite subcover of C is a finite subcollection whose union still contains X. To say that X is *compact* means that every open cover of X has a finite subcover. Similarly, if $Z \subseteq X$ then an open cover C of Z is a collection of open subsets whose union contains Z. We say that Z is compact if every open cover of Z has a finite subcover. If X is compact, then it is easy to show that any closed subset of X is also compact.

A topological space Y is *Hausdorff* if, for any two distinct points y_1, y_2 in Y, there are disjoint open sets $G_1, G_2 \subseteq Y$ with $y_1 \in G_1$ and $y_2 \in G_2$. It is not hard to show that any compact subset of a Hausdorff space is closed.

If X and Y are topological spaces then a map $\theta\colon X \to Y$ is *continuous* if, for all open sets $G \subseteq Y$, the set $\theta^{-1}(G)$ is open in X. Taking complements, we see that θ is continuous if and only if, for all closed $K \subseteq Y$, the set $\theta^{-1}(K)$ is closed in X. If Z is a compact subset of X and $\theta\colon X \to Y$ is continuous, then $\theta(Z)$ is compact.

A *homeomorphism* from X to Y is a bijection $X \to Y$ which is continuous and has a continuous inverse. If there is a homeomorphism from X to Y, we say that X and Y are *homeomorphic*. If X and Y are homeomorphic topological spaces then all of their topological properties are identical. In particular, X is compact if and only if Y is compact.

Recall that if X is a topological space and $Z \subseteq X$, then the *subspace topology* on Z is defined by declaring the open sets of Z to be the sets $G \cap Z$ for G an open set of X. Then Z is a compact subset of X if and only if Z is a compact topological space (in the subspace topology). Also, it is easy to see that a subspace of a Hausdorff space is Hausdorff.

2.1.1 Lemma. *Let X and Y be topological spaces, and suppose that X is compact and Y is Hausdorff.*

(i). If $\theta\colon X \to Y$ is a continuous bijection, then θ is a homeomorphism onto Y. In particular, Y is compact.

(ii). If $\theta\colon X \to Y$ is a continuous injection, then $\theta(X)$ (with the subspace topology from Y) is homeomorphic to X. In particular, $\theta(X)$ is a compact subset of Y.

Proof. (i) Let K be a closed subset of X. Since X is compact, K is compact. Since θ is continuous, $\theta(K)$ is compact. A compact subset of a Hausdorff space is closed, so $\theta(K)$ is closed. Hence θ^{-1} is continuous, which shows that θ is a homeomorphism. So Y is homeomorphic to the compact space X; so Y is compact.

(ii) The subspace $\theta(X)$ of the Hausdorff space Y is Hausdorff. Let $\widetilde{\theta}\colon X \to \theta(X)$, $x \mapsto \theta(x)$, which is a continuous bijection. By (i), $\widetilde{\theta}$ is a homeomorphism and $\theta(X)$ is compact. □

2.2 Subbases and weak topologies

2.2.1 Definition. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are two topologies on a set X then we say that $\mathcal{T}_1$ is *weaker* than $\mathcal{T}_2$ if $\mathcal{T}_1 \subseteq \mathcal{T}_2$.
[We might also say that $\mathcal{T}_1$ is *smaller*, or *coarser* than $\mathcal{T}_2$].

2.2.2 Definition. Let $(X, \mathcal{T})$ be a topological space. We say that a collection of open sets $\mathcal{S} \subseteq \mathcal{T}$ is a *subbase* for $\mathcal{T}$ if $\mathcal{T}$ is the weakest topology containing $\mathcal{S}$. If the topology $\mathcal{T}$ is understood, we will also say that $\mathcal{S}$ is a subbase for the topological space X.

2.2.3 Remark. Suppose that $\mathcal{T}$ is a topology on X and $\mathcal{S} \subseteq \mathcal{T}$. It is not hard to see that the collection of unions of finite intersections of sets in $\mathcal{S}$ forms a topology on X which is no larger than $\mathcal{T}$. [The empty set is the union of zero sets, and X is the intersection of zero sets, so $\emptyset$ and X are in this collection.] From this, it follows that that following conditions are equivalent:

(i). $\mathcal{S}$ is a subbase for $\mathcal{T}$;

(ii). every set in $\mathcal{T}$ is a union of finite intersections of sets in $\mathcal{S}$;

(iii). a set $G \subseteq X$ is in $\mathcal{T}$ if and only if for every $x \in G$, there exist finitely many sets $S_1, S_2, \ldots, S_n \in \mathcal{S}$ such that

$$x \in S_1 \cap S_2 \cap \cdots \cap S_n \subseteq G.$$

By the equivalence of (i) and (ii), if X, Y are topological spaces and $\mathcal{S}$ is a subbase for X, then a map $f\colon Y \to X$ is continuous if and only if $f^{-1}(S)$ is open for all $S \in \mathcal{S}$.

2.2.4 Proposition. *Let X be a set, let I be an index set and suppose that X_i is a topological space and $f_i\colon X \to X_i$ for each $i \in I$. The collection*

$$\mathcal{S} = \{f_i^{-1}(G)\colon i \in I,\ G \text{ is an open subset of } X_i\}$$

is a subbase for a topology on X, and this is the weakest topology such that f_i is continuous for all $i \in I$.

Proof. Let $\mathcal{T}$ be the collection of all unions of finite intersections of sets from $\mathcal{S}$. Then $\mathcal{T}$ is a topology on X and $\mathcal{S}$ is a subbase for $\mathcal{T}$ by the previous remark. If $\mathcal{T}'$ is any topology on X such that $f_i\colon X \to X_i$ is continuous for all $i \in I$ then by the definition of continuity, $f_i^{-1}(G) \in \mathcal{T}'$ for all open subsets $G \subseteq X_i$, so $\mathcal{S} \subseteq \mathcal{T}'$. Since $\mathcal{T}$ is the weakest topology containing $\mathcal{S}$ this shows that $\mathcal{T} \subseteq \mathcal{T}'$, so $\mathcal{T}$ is the weakest topology such that each f_i is continuous. □

This allows us to introduce the following terminology.

2.2.5 Definition. If X is a set, X_i is a topological space and $f_i\colon X \to X_i$ for $i \in I$ then the weakest topology on X such that f_i is continuous for each $i \in I$ is called the *weak topology* induced by the family $\{f_i\colon i \in I\}$.

2.2.6 Proposition. *Suppose that X is a topological space with the weak topology induced by a family of maps $\{f_i\colon i \in I\}$ where $f_i\colon X \to X_i$ and X_i is a topological space for each $i \in I$.*

If Y is a topological space then a map $g\colon Y \to X$ is continuous if and only if $f_i \circ g\colon Y \to X_i$ is continuous for all $i \in I$.

Proof. The sets $f_i^{-1}(G)$ for $i \in I$ and G an open subset of X_i form a subbase for X, by Proposition 2.2.4. Hence, by Remark 2.2.3,

$$\begin{aligned} g \text{ is continuous} &\iff g^{-1}(f_i^{-1}(G)) \text{ is open for all } i \in I \text{ and open } G \subseteq X_i \\ &\iff (f_i \circ g)^{-1}(G) \text{ is open for all } i \in I \text{ and open } G \subseteq X_i \\ &\iff f_i \circ g \text{ is continuous for all } i \in I. \end{aligned}$$

□

2.2.7 Lemma. *Suppose that X is a topological space with the weak topology induced by a family of mappings $\{f_i\colon X \to X_i\}_{i \in I}$. If $Y \subseteq X$ then the weak topology induced by the family $\{f_i|_Y\colon Y \to X_i\}_{i \in I}$ is the subspace topology on Y.*

Proof. Let $g_i = f_i|_Y$ for $i \in I$. Observe that $g_i^{-1}(G) = f_i^{-1}(G) \cap Y$ for G an open subset of X_i. It is not hard to check that the collection of all sets of this form is a subbase for both the weak topology induced by $\{g_i\}_{i \in I}$ and for the subspace topology on Y. Hence these topologies are equal. □

2.3 The product topology and Tychonoff's theorem

If $\mathcal{S}$ is a collection of open subsets of X, let us say that an open cover is an $\mathcal{S}$*-cover* if every set in the cover is in $\mathcal{S}$.

2.3.1 Theorem (Alexander's subbase lemma)**.** *Let X be a topological space with a subbase $\mathcal{S}$. If every $\mathcal{S}$-cover of X has a finite subcover, then X is compact.*

Proof. Suppose that the hypothesis holds but that X is not compact. Then there is an open cover with no finite subcover; ordering such covers by inclusion we can apply Zorn's lemma [FA 2.15] to find an open cover C of X without a finite subcover that is maximal among such covers.

Note that $C \cap \mathcal{S}$ cannot cover X by hypothesis, so there is some $x \in X$ which does not lie in any set in the collection $C \cap \mathcal{S}$. On the other hand, C does cover X so there is some $G \in C \setminus \mathcal{S}$ with $x \in G$. Since G is open and $\mathcal{S}$ is a subbase, by Remark 2.2.3 we have

$$x \in S_1 \cap \cdots \cap S_n \subseteq G$$

for some $S_1, \dots, S_n \in \mathcal{S}$. For $i = 1, \dots, n$ we have $x \in S_i$ so $S_i \notin C$. By the maximality of C, there is a finite subcover C_i of $C \cup \{S_i\}$. Let $D_i = C_i \setminus \{S_i\}$. Then $D = D_1 \cup \cdots \cup D_n$ covers $X \setminus (S_1 \cap \cdots \cap S_n)$. So $D \cup \{G\}$ is a finite subcover of C, which is a contradiction. So X must be compact. □

2.3.2 Definition. Let $\{X_i \colon i \in I\}$ be an indexed collection of sets. Just as in [FA 2.4], we define the *(Cartesian) product* of this collection to be the set

$$\prod_{i \in I} X_i = \{f \colon I \to \bigcup_{i \in I} X_i \colon f(i) \in X_i \text{ for all } i \in I\}.$$

If each X_i is a topological space then the *product topology* on $X = \prod_{i \in I} X_i$ is the weak topology induced by the family $\{\pi_i \colon i \in I\}$ where the map π_i is the "evaluation at i" map

$$\pi_i \colon X \to X_i, \quad f \mapsto f(i).$$

2.3.3 Remark. If $f \in \prod_{i \in I} X_i$ then it is often useful to think of f as the "I-tuple" $(f(i))_{i \in I}$. In this notation, we have

$$\prod_{i \in I} X_i = \{(x_i)_{i \in I} \colon x_i \in X_i \text{ for all } i \in I\}$$

and $\pi_i \colon (x_i)_{i \in I} \mapsto x_i$ is the projection onto the ith coordinate.

2.3.4 Theorem (Tychonoff's theorem)**.**
The product of a collection of compact topological spaces is compact.

Proof. Let X_i be a compact topological space for $i \in I$ and let $X = \prod_{i \in I} X_i$ with the product topology. Consider the collection

$$\mathcal{S} = \{\pi_i^{-1}(G) \colon i \in I,\ G \text{ is an open subset of } X_i\}.$$

By Proposition 2.2.4, $\mathcal{S}$ is a subbase for the topology on X.

Let C be an $\mathcal{S}$-cover of X. For $i \in I$, let $C_i = \{G \subseteq X_i \colon \pi_i^{-1}(G) \in C\}$, which is a collection of open subsets of X_i.

We claim that there is $i \in I$ such that C_i is a cover of X_i. Otherwise, for every $i \in I$ there is some $x_i \in X_i$ not covered by C_i. Consider the map $f \in X$ defined by $f(i) = x_i$. By construction, f does not lie in $\pi_i^{-1}(G)$ for any $i \in I$ and $G \in C_i$. However, since $C \subseteq \mathcal{S}$, every set in C is of this form, so C cannot cover f. This contradiction establishes the claim.

So we can choose $i \in I$ so that C_i covers X_i. Since X_i is compact, there is finite subcover D_i of C_i. But then $\{\pi_i^{-1}(G) \colon G \in D_i\}$ covers X, and this is a finite subcover of C.

This shows that every $\mathcal{S}$-cover of X has a finite subcover. By Theorem 2.3.1, X is compact. □

2.4 The weak* topology

Let X be a Banach space. Recall from [FA 3.2] that the dual space X^* of X is the Banach space of continuous linear functionals $\varphi : X \to \mathbb{C}$, with the norm $\|\varphi\| = \sup\limits_{\|x\| \leq 1} |\varphi(x)|$.

2.4.1 Definition. For $x \in X$, let $J_x \colon X^* \to \mathbb{C}$, $\varphi \mapsto \varphi(x)$. The *weak* topology* on X^* is the weak topology induced by the family $\{J_x \colon x \in X\}$.

2.4.2 Remarks. (i). For $x \in X$, the map J_x is simply the canonical image of x in X^{**}. In particular, each J_x is continuous when X^* is equipped with the usual topology from its norm, so the weak* topology is weaker (that is, no stronger) than the norm topology on X^*. In fact, the weak* topology is generally strictly weaker than the norm topology.

(ii). The sets $\{\psi \in X^* \colon |\psi(x) - \varphi(x)| < \varepsilon\}$ for $\varphi \in X^*$, $\varepsilon > 0$ and $x \in X$ form a subbase for the weak* topology.

(iii). By (ii), it is easy to see that X^* with the weak* topology is a Hausdorff topological space.

2.4.3 Theorem (The Banach-Alaoglu theorem). *Let X be a Banach space. The closed unit ball of X^* is compact in the weak* topology.*

Proof. For $x \in X$ let $D_x = \{\lambda \in \mathbb{C} : |\lambda| \leq \|x\|\}$. Since D_x is closed and bounded, it is a compact topological subspace of $\mathbb{C}$. Let $D = \prod_{x \in X} D_x$ with the product topology. By Tychonoff's theorem 2.3.4, D is a compact topological space.

Let $X_1^* = \{\varphi \in X^* : \|\varphi\| \leq 1\}$ denote the closed unit ball of X^*, with the subspace topology that it inherits from the weak* topology on X^*. We must show that X_1^* is compact.

If φ is any linear map $X \to \mathbb{C}$, then $\varphi \in X_1^*$ if and only if $|\varphi(x)| \leq \|x\|$ for all $x \in X$, i.e. $\varphi(x) \in D_x$ for all $x \in X$. Thus $X_1^* \subseteq D$. Moreover, if $x \in X$ and $\varphi \in X_1^*$ then

$$J_x(\varphi) = \varphi(x) = \pi_x(\varphi),$$

so $J_x|_{X_1^*} = \pi_x|_{X_1^*}$. By Lemma 2.2.7, the topology on X_1^* is equal to the subspace topology when we view it as a subspace of D.

Since D is compact, it suffices to show that X_1^* is a closed subset of D. Now

$$X_1^* = \{\varphi \in D : \varphi \text{ is linear}\} = \bigcap_{\substack{\alpha,\beta \in \mathbb{C}, \\ x,y \in X}} \{\varphi \in D : \pi_{\alpha x + \beta y}(\varphi) = \alpha \pi_x(\varphi) + \beta \pi_y(\varphi)\}.$$

By the definition of the product topology on D, the maps $\pi_x : D \to D_x$ are continuous for each $x \in X$. Linear combinations of continuous functions are continuous, so for $x, y, z \in X$ and $\alpha, \beta \in \mathbb{C}$, the function $\rho = \pi_z - \alpha\pi_x - \beta\pi_y$ is continuous $D \to \mathbb{C}$. Hence $\rho^{-1}(0)$ is closed. Each set in the above intersection is of this form, so X_1^* is closed. $\square$