442C Banach algebras 2009–10

1 Introduction to Banach algebras

1.1 Definitions and examples

Let us adopt the convention that all vector spaces and Banach spaces are over the field of complex numbers.

1.1.1 Definition. A *Banach algebra* is a vector space A such that

(i). A is an algebra: it is equipped with an associative product $(a, b) \mapsto ab$ which is linear in each variable,

(ii). A is a Banach space: it has a norm $\|\cdot\|$ with respect to which it is complete, and

(iii). A is a normed algebra: we have $\|ab\| \leq \|a\| \, \|b\|$ for $a, b \in A$.

If the product is commutative, so that $ab = ba$ for all $a, b \in A$, then we say that A is an *abelian* Banach algebra.

1.1.2 Remark. The inequality $\|ab\| \leq \|a\| \, \|b\|$ ensures that the product is continuous as a map $A \times A \to A$.

1.1.3 Examples. (i). Let X be a topological space. We write $BC(X)$ for the set of bounded continuous functions $X \to \mathbb{C}$. Recall from [FA 1.7.2] that $BC(X)$ is a Banach space under the pointwise vector space operations and the uniform norm, which is given by

$$\|f\| = \sup_{x \in X} |f(x)|, \quad f \in BC(X).$$

The pointwise product

$$(fg)(x) = f(x)g(x), \quad f, g \in BC(X), \; x \in X$$

turns $BC(X)$ into an abelian Banach algebra. Indeed, the product is clearly commutative and associative, it is linear in f and g, and

$$\|fg\| = \sup_{x \in X} |f(x)| \, |g(x)| \leq \sup_{x_1 \in X} |f(x_1)| \cdot \sup_{x_2 \in X} |g(x_2)| = \|f\| \, \|g\|.$$

If X is a compact space, then every continuous function $X \to \mathbb{C}$ is bounded. For this reason, we will write $C(X)$ instead of $BC(X)$ if X is compact.

(ii). If A is a Banach algebra, a *subalgebra* of A is a linear subspace $B \subseteq A$ such that $a, b \in B \implies ab \in B$. If B is a closed subalgebra of a Banach algebra A, then B is complete, so it is a Banach algebra (under the same operations and norm as A). We then say that B is a *Banach subalgebra* of A.

(iii). Let $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$ and $\overline{\mathbb{D}} = \{z \in \mathbb{C} \colon |z| \leq 1\}$. The *disc algebra* is the following Banach subalgebra of $C(\overline{\mathbb{D}})$:

$$A(\overline{\mathbb{D}}) = \{f \in C(\overline{\mathbb{D}}) \colon f \text{ is analytic on } \mathbb{D}\}.$$

(iv). The set $C_0(\mathbb{R})$ of continuous functions $f \colon \mathbb{R} \to \mathbb{C}$ such that

$$\lim_{x \to \infty} f(x) = \lim_{x \to -\infty} f(x) = 0.$$

This is a Banach subalgebra of $BC(\mathbb{R})$.

(v). Let $\ell^1(\mathbb{Z})$ denote the vector space of complex sequences $(a_n)_{n \in \mathbb{Z}}$ indexed by $\mathbb{Z}$ such that $\|a\| = \sum_{n \in \mathbb{Z}} |a_n| < \infty$. This is a Banach space by [FA 1.7.10]. We define a product $*$ such that if $a = (a_n)_{n \in \mathbb{Z}}$ and $b = (b_n)_{n \in \mathbb{Z}}$ are in $\ell^1(\mathbb{Z})$ then the nth entry of $a * b$ is

$$(a * b)_n = \sum_{m \in \mathbb{Z}} a_m b_{n-m}.$$

This series is absolutely convergent, and $a * b \in \ell^1(\mathbb{Z})$, since

$$\begin{aligned} \|a * b\| &= \sum_n |(a * b)_n| = \sum_n \Big| \sum_m a_m b_{n-m} \Big| \\ &\leq \sum_{m,n} |a_m| \, |b_{n-m}| = \sum_m |a_m| \sum_n |b_{m-n}| = \|a\| \, \|b\| < \infty. \end{aligned}$$

It is an exercise to show that $*$ is commutative, associative and linear in each variable, so it turns $\ell^1(\mathbb{Z})$ into an abelian Banach algebra.

(vi). If X is a Banach space, let $\mathcal{B}(X)$ denote the set of all bounded linear operators $T \colon X \to X$ with the operator norm

$$\|T\| = \sup_{x \in X, \, \|x\| \leq 1} \|Tx\|.$$

By [FA 3.3], $\mathcal{B}(X)$ is a Banach space. Define a product on $\mathcal{B}(X)$ by $ST = S \circ T$. This is clearly associative and bilinear, and if $x \in X$ with $\|x\| \leq 1$ then

$$\|(ST)x\| = \|S(Tx)\| \leq \|S\| \, \|Tx\| \leq \|S\| \, \|T\|$$

so $\|ST\| \leq \|S\| \, \|T\|$. Hence $\mathcal{B}(X)$ is a Banach algebra. If $\dim X > 1$ then $\mathcal{B}(X)$ is not abelian.

1.2 Invertibility

1.2.1 Definition. A Banach algebra A is *unital* if A contains an identity element of norm 1; that is, an element $1 \in A$ such that $1a = a1 = a$ for all $a \in A$, and $\|1\| = 1$. We call 1 the *unit* of A. We sometimes write $1 = 1_A$ to make it clear that 1 is the unit of A.

If A is a unital Banach algebra and $B \subseteq A$, we say that B is a *unital subalgebra* of A if B is a subalgebra of A which contains the unit of A.

1.2.2 Examples. Of the Banach algebras in Example 1.1.3, only $C_0(\mathbb{R})$ is non-unital. Indeed, it is easy to see that no $f \in C_0(\mathbb{R})$ is an identity element for $C_0(\mathbb{R})$. On the other hand the constant function taking the value 1 is the unit for $BC(X)$, and $A(\overline{\mathbb{D}})$ is a unital subalgebra of $C(\overline{\mathbb{D}})$. Also, the identity operator $I\colon X \to X$, $x \to x$ is the unit for $\mathcal{B}(X)$, and is it not hard to check that the sequence $(\delta_{n,0})_{n\in\mathbb{Z}}$ is the unit for $\ell^1(\mathbb{Z})$.

1.2.3 Remarks. (i). An algebra can have at most one identity element.

(ii). If $(A, \|\cdot\|)$ is a non-zero Banach algebra with an identity element then we can define an norm $|\cdot|$ on A under which it is a unital Banach algebra such that $|\cdot|$ is equivalent to $\|\cdot\|$, meaning that there are constants $m, M \geq 0$ such that

$$m|a| \leq \|a\| \leq M|a| \quad \text{for all } a \in A.$$

For example, we could take $|a| = \|L_a\colon A \to A\|$ where L_a is the linear operator $L_a(b) = ab$ for $a, b \in A$.

1.2.4 Definition. Let A be a Banach algebra with unit 1. An element $a \in A$ is *invertible* if $ab = 1 = ba$ for some $b \in A$. It's easy to see that b is then unique; we call b the *inverse* of a and write $b = a^{-1}$.

We write $\operatorname{Inv} A$ for the set of invertible elements of A.

1.2.5 Remarks. (i). $\operatorname{Inv} A$ forms a group under multiplication.

(ii). If $a \in A$ is left invertible and right invertible so that $ba = 1$ and $ac = 1$ for some $b, c \in A$, then a is invertible.

(iii). If $a = bc = cb$ then a is invertible if and only if b and c are invertible. It follows by induction that if $b_1, \dots, b_n$ are commuting elements of A (meaning that $b_i b_j = b_j b_i$ for $1 \leq i, j \leq n$) then $b_1 b_2 \dots b_n$ is invertible if and only if $b_1, \dots, b_n$ are all invertible.

(iv). The commutativity hypothesis is essential in (iii). For example, if $(e_n)_{n\geq 1}$ is an orthonormal basis of a Hilbert space H and $S \in \mathcal{B}(H)$ is defined by $Se_n = e_{n+1}$, $n \geq 1$ and S^* is the adjoint of S (see [FA 4.18]) then S^*S is invertible although S^* and S are not.

1.2.6 Examples. (i). If X is a compact topological space then

$$\operatorname{Inv} C(X) = \{f \in C(X) \colon f(x) \neq 0 \text{ for all } x \in X\}.$$

Indeed, if $f(x) \neq 0$ for all $x \in X$ then we can define $g \colon X \to \mathbb{C}$, $x \mapsto f(x)^{-1}$. The function g is then continuous [why?] with $fg = 1$. Conversely, if $x \in X$ and $f(x) = 0$ then $fg(x) = f(x)g(x) = 0$ so $fg \neq 1$ for all $g \in C(X)$, so f is not invertible.

(ii). If X is a Banach space then

$$\operatorname{Inv} \mathcal{B}(X) \subseteq \big\{T \in \mathcal{B}(X) \colon \ker T = \{0\}\big\}.$$

Indeed, if $\ker T \neq \{0\}$ then T is not injective, so cannot be invertible.

If X is finite-dimensional and $\ker T = \{0\}$ then by linear algebra, T is surjective, so T is an invertible linear map. Since X is finite-dimensional, the linear map T^{-1} is bounded, so T is invertible in $\mathcal{B}(X)$. Hence

$$\operatorname{Inv} \mathcal{B}(X) = \big\{T \in \mathcal{B}(X) \colon \ker T = \{0\}\big\} \quad \text{if } \dim X < \infty.$$

On the other hand, if X is infinite-dimensional then we generally have $\operatorname{Inv} \mathcal{B}(X) \subsetneq \big\{T \in \mathcal{B}(X) \colon \ker T = \{0\}\big\}$. For example, let $X = H$ be an infinite-dimensional Hilbert space with orthonormal basis $(e_n)_{n \geq 1}$. Consider the operator $T \in \mathcal{B}(H)$ defined by

$$Te_n = \tfrac{1}{n} e_n, \quad n \geq 1.$$

It is easy to see that $\ker T = \{0\}$. However, T is not invertible. Indeed, if $S \in \mathcal{B}(H)$ with $ST = I$ then $Se_n = S(nTe_n) = nSTe_n = ne_n$, so

$$\|Se_n\| = n \to \infty \text{ as } n \to \infty$$

and so S is not bounded, which is a contradiction.

1.2.7 Theorem. *Let A be a Banach algebra with unit 1. If $a \in A$ with $\|a\| < 1$ then $1 - a \in \operatorname{Inv} A$ and*

$$(1-a)^{-1} = \sum_{n=0}^{\infty} a^n.$$

Proof. Since $\|a^n\| \leq \|a\|^n$ and $\|a\| < 1$, the series $\sum_{n=0}^{\infty} a^n$ is absolutely convergent and so convergent by [FA 1.7.8], say to $b \in A$. Let b_n be the nth partial sum of this series and note that

$$b_n(1-a) = (1-a)b_n = (1-a)(1 + a + a^2 + \cdots + a^n) = 1 - a^{n+1} \to 1$$

as $n \to \infty$. So $b(1-a) = (1-a)b = 1$ and so $b = (1-a)^{-1}$. $\square$

1.2.8 Corollary. $\operatorname{Inv} A$ *is an open subset of* A.

Proof. Let $a \in \operatorname{Inv} A$ and let $r_a = \|a^{-1}\|^{-1}$. We claim that the open ball $B(a, r_a) = \{b \in A \colon \|a - b\| < r_a\}$ is contained in $\operatorname{Inv} A$; for if $b \in B(a, r_a)$ then $\|a - b\| < r_a$ and

$$b = (a - (a - b))a^{-1}a = (1 - (a - b)a^{-1})a.$$

Since $\|(a - b)a^{-1}\| < r_a\|a^{-1}\| < 1$, the element $1 - (a - b)a^{-1}$ is invertible by Theorem 1.2.7. Hence b is the product of two invertible elements, so is invertible. This shows that every element of $\operatorname{Inv} A$ may be surrounded by an open ball which is contained in $\operatorname{Inv} A$, hence $\operatorname{Inv} A$ is open. $\square$

1.2.9 Corollary. *The map* $\theta \colon \operatorname{Inv} A \to \operatorname{Inv} A$, $a \mapsto a^{-1}$ *is a homeomorphism.*

Proof. Since $(a^{-1})^{-1} = a$, the map θ is a bijection with $\theta = \theta^{-1}$. So we only need to show that θ is continuous.

If $a \in \operatorname{Inv} A$ and $b \in \operatorname{Inv} A$ with $\|a - b\| < \frac{1}{2}\|a^{-1}\|^{-1}$ then using the triangle inequality and the identity

$$a^{-1} - b^{-1} = a^{-1}(b - a)b^{-1} \qquad (\bigstar)$$

we have

$$\|b^{-1}\| \le \|a^{-1} - b^{-1}\| + \|a^{-1}\| \le \|a^{-1}\|\,\|a - b\|\,\|b^{-1}\| + \|a^{-1}\| \le \tfrac{1}{2}\|b^{-1}\| + \|a^{-1}\|,$$

so $\|b^{-1}\| \le 2\|a^{-1}\|$. Using $(\bigstar)$ again, we have

$$\|\theta(a) - \theta(b)\| = \|a^{-1} - b^{-1}\| \le \|a^{-1}\|\,\|a - b\|\,\|b^{-1}\| < 2\|a^{-1}\|^2\|a - b\|,$$

which shows that θ is continuous at a. $\square$

1.3 The spectrum

1.3.1 Definition. Let A be a unital Banach algebra and let $a \in A$. The *spectrum* of a in A is

$$\sigma(a) = \sigma_A(a) = \{\lambda \in \mathbb{C} \colon \lambda 1 - a \notin \operatorname{Inv} A\}.$$

We will often write λ instead of $\lambda 1$ for $\lambda \in \mathbb{C}$.

1.3.2 Examples. (i). We have $\sigma(\lambda 1) = \{\lambda\}$ for any $\lambda \in \mathbb{C}$.

(ii). Let X be a compact topological space. If $f \in C(X)$ then

$$\sigma(f) = f(X) = \{f(x)\colon x \in X\}.$$

Indeed,

$$\begin{aligned} \lambda \in \sigma(f) &\iff \lambda 1 - f \notin \operatorname{Inv} C(X) \\ &\iff (\lambda 1 - f)(x) = 0 \text{ for some } x \in X, \text{ by Example 1.2.6(i)} \\ &\iff \lambda = f(x) \text{ for some } x \in X \\ &\iff \lambda \in f(X). \end{aligned}$$

(iii). If X is a finite-dimensional Banach space and $T \in \mathcal{B}(X)$ then

$$\sigma(T) = \{\lambda \in \mathbb{C}\colon \lambda \text{ is an eigenvalue of } T\}.$$

Indeed,

$$\begin{aligned} \lambda \in \sigma(T) &\iff T - \lambda \notin \operatorname{Inv} \mathcal{B}(X) \\ &\iff \ker(\lambda I - T) \neq \{0\} \text{ by Example 1.2.6(ii)} \\ &\iff (\lambda I - T)(x) = 0 \text{ for some nonzero } x \in X \\ &\iff Tx = \lambda x \text{ for some nonzero } x \in X \\ &\iff \lambda \text{ is an eigenvalue of } T. \end{aligned}$$

If X is an infinite-dimensional Banach space, then the same argument shows that $\sigma(T)$ contains the eigenvalues of T, but generally this inclusion is strict.

We will need the following algebraic fact later on.

1.3.3 Proposition. *Let A be a unital Banach algebra and let $a, b \in A$.*

(i). If $1 - ab \in \operatorname{Inv} A$ then $1 - ba \in \operatorname{Inv} A$, and

$$(1 - ba)^{-1} = 1 + b(1 - ab)^{-1}a.$$

(ii). $\sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\}$.

Proof. Exercise. □

To show that the spectrum is always non-empty, we will use a vector-valued version of Liouville's theorem:

1.3.4 Lemma. *Let X be a Banach space and suppose that $f\colon \mathbb{C} \to X$ is an entire function in the sense that $\frac{f(\mu)-f(\lambda)}{\mu-\lambda}$ converges in X as $\mu \to \lambda$, for every $\lambda \in \mathbb{C}$. If f is bounded then f is constant.*

Proof. Given a continuous linear functional $\varphi \in X^*$, let $g = \varphi \circ f\colon \mathbb{C} \to \mathbb{C}$. Since $\frac{g(\mu)-g(\lambda)}{\mu-\lambda} = \varphi(\frac{f(\mu)-f(\lambda)}{\mu-\lambda})$ and $|g(\lambda)| \leq \|g\|\,\|f(\lambda)\|$, the function g is entire and bounded. By Liouville's theorem it is constant, so $\varphi(f(\lambda)) = \varphi(f(\mu))$ for all $\varphi \in X^*$ and $\lambda, \mu \in \mathbb{C}$. By the Hahn-Banach theorem (see [FA 3.8]), $f(\lambda) = f(\mu)$ for all $\lambda, \mu \in \mathbb{C}$. So f is constant. $\square$

1.3.5 Theorem. *Let A be a unital Banach algebra. If $a \in A$ then $\sigma(a)$ is a non-empty compact subset of $\mathbb{C}$ with $\sigma(a) \subseteq \{\lambda \in \mathbb{C}\colon |\lambda| \leq \|a\|\}$.*

Proof. The map $i\colon \mathbb{C} \to A$, $\lambda \mapsto \lambda - a$ is continuous and

$$\sigma(a) = \{\lambda \in \mathbb{C}\colon i(\lambda) \notin \operatorname{Inv} A\} = \mathbb{C} \setminus i^{-1}(\operatorname{Inv} A).$$

Since $\operatorname{Inv} A$ is open by Corollary 1.2.8 and i is continuous, $i^{-1}(\operatorname{Inv} A)$ is open and so its complement $\sigma(a)$ is closed.

If $|\lambda| > \|a\|$ then $\lambda - a = \lambda(1-\lambda^{-1}a)$ and $\|\lambda^{-1}a\| = |\lambda|^{-1}\|a\| < 1$ so $\lambda - a$ is invertible by Theorem 1.2.7, so $\lambda \notin \sigma(a)$. Hence $\sigma(a) \subseteq \{\lambda \in \mathbb{C}\colon |\lambda| \leq \|a\|\}$. In particular, $\sigma(a)$ is bounded as well as closed, so $\sigma(a)$ is a compact subset of $\mathbb{C}$.

Finally, we must show that $\sigma(a) \neq \emptyset$. If $\sigma(a) = \emptyset$ then the map

$$R\colon \mathbb{C} \to A, \quad \lambda \mapsto (\lambda - a)^{-1}$$

is well-defined. It not hard to show using (★) that

$$\frac{R(\mu) - R(\lambda)}{\mu - \lambda} = -R(\lambda)R(\mu) \quad \text{for } \lambda, \mu \in \mathbb{C} \text{ with } \lambda \neq \mu.$$

Corollary 1.2.9 shows that R is continuous, so we conclude that R is an entire function (with derivative $R'(\lambda) = -R(\lambda)^2$).

Now $\|R(\lambda)\| = \|(\lambda - a)^{-1}\| = |\lambda|^{-1}\|(1 - \lambda^{-1}a)^{-1}\|$ and $1 - \lambda^{-1}a \to 1$ as $|\lambda| \to \infty$ so, by Corollary 1.2.9, $(1 - \lambda^{-1}a)^{-1} \to 1$. Hence $\|R(\lambda)\| \to 0$ as $|\lambda| \to \infty$.

Hence R is a bounded entire function, so it is constant by Lemma 1.3.4; since $R(\lambda) \to 0$ as $|\lambda| \to \infty$ we have $R(\lambda) = 0$ for all $\lambda \in \mathbb{C}$. This is a contradiction since $R(\lambda)$ is invertible for any $\lambda \in \mathbb{C}$. $\square$

The next result says that $\mathbb{C}$ is essentially the only unital Banach algebra which is also a field.

1.3.6 Corollary (The Gelfand-Mazur theorem)**.** *If A is a unital Banach algebra in which every non-zero element is invertible then $A = \mathbb{C}1_A$.*

Proof. Let $a \in A$. Since $\sigma(a) \neq \emptyset$ there is some $\lambda \in \sigma(a)$. Now $\lambda 1 - a \notin \operatorname{Inv} A$, so $\lambda 1 - a = 0$ and $a = \lambda 1 \in \mathbb{C}1$. □

1.3.7 Definition. If a is an element of a unital Banach algebra and $p \in \mathbb{C}[z]$ is a complex polynomial, say $p = \lambda_0 + \lambda_1 z + \cdots + \lambda_n z^n$ where $\lambda_0, \lambda_1, \ldots, \lambda_n$ are complex numbers, then we write

$$p(a) = \lambda_0 1 + \lambda_1 a + \cdots + \lambda_n a^n.$$

1.3.8 Theorem (The spectral mapping theorem for polynomials)**.** *If p is a complex polynomial and a is an element of a unital Banach algebra then*

$$\sigma(p(a)) = p(\sigma(a)) = \{p(\lambda)\colon \lambda \in \sigma(a)\}.$$

Proof. If p is a constant then this immediate since $\sigma(\lambda 1) = \{\lambda\}$. Suppose that $n = \deg p \geq 1$ and let $\mu \in \mathbb{C}$. Since $\mathbb{C}$ is algebraically closed, we can write

$$\mu - p = C(\lambda_1 - z) \ldots (\lambda_n - z)$$

for some $C, \lambda_1, \ldots, \lambda_n \in \mathbb{C}$. Then

$$\mu - p(a) = C(\lambda_1 - a) \ldots (\lambda_n - a)$$

and the factors $\lambda_i - a$ all commute. So

$$\begin{aligned}
\mu \in \sigma(p(a)) &\iff \mu - p(a) \text{ is not invertible} \\
&\iff \text{some } \lambda_i - a \text{ is not invertible (by Remark 1.2.5(iii))} \\
&\iff \text{some } \lambda_i \text{ is in } \sigma(a) \\
&\iff \sigma(a) \text{ contains a root of } \mu - p \\
&\iff \mu = p(\lambda) \text{ for some } \lambda \in \sigma(a).
\end{aligned}$$

□

1.3.9 Definition. Let A be a unital Banach algebra. The *spectral radius* of an element $a \in A$ is

$$r(a) = r_A(x) = \sup_{\lambda \in \sigma_A(a)} |\lambda|.$$

1.3.10 Remark. We have $r(a) \leq \|a\|$ by Theorem 1.3.5.

1.3.11 Examples. (i). If X is a compact topological space and $f \in C(X)$, then

$$r(f) = \sup_{\lambda \in \sigma_A(f)} |\lambda| = \sup_{\lambda \in f(X)} |\lambda| = \|f\|.$$

(ii). To see that strict inequality is possible, take $X = \mathbb{C}^2$ with the usual Hilbert space norm and let $T \in \mathcal{B}(X)$ be the operator with matrix $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Since $\det(T - \lambda I) = \lambda^2$, the only eigenvalue of T is 0 and so $\sigma(T) = \{0\}$ by Example 1.3.2(iii). Hence $r(T) = 0 < 1 = \|T\|$.

1.3.12 Theorem (The spectral radius formula)**.** *The spectral radius of an element of a unital Banach algebra is given by*

$$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n} = \inf_{n\geq 1} \|a^n\|^{1/n}.$$

Proof. If $\lambda \in \sigma(a)$ and $n \geq 1$ then $\lambda^n \in \sigma(a^n)$ by Theorem 1.3.8. So $|\lambda|^n \leq \|a^n\|$ by Theorem 1.3.5, hence $|\lambda| \leq \|a^n\|^{1/n}$ and so $r(a) \leq \inf_{n\geq 1} \|a^n\|^{1/n}$.

Consider the function

$$S\colon \{\lambda \in \mathbb{C}\colon |\lambda| < 1/r(a)\} \to A, \quad \lambda \mapsto (1 - \lambda a)^{-1}.$$

Observe that for $|\lambda| < 1/r(a)$ we have $r(\lambda a) = |\lambda| r(a) < 1$ by Theorem 1.3.8, so $1 - \lambda a$ is invertible and $S(\lambda)$ is well-defined. We can argue as in the proof of Theorem 1.3.5 to see that S is holomorphic. By Theorem 1.2.7 we have $S(\lambda) = \sum_{n=0}^{\infty} \lambda^n a^n$ for $|\lambda| < 1/\|a\|$. If $\varphi \in A^*$ with $\|\varphi\| = 1$ then the complex-valued function $f = \varphi \circ S$ is given by the power series $f(\lambda) = \sum_{n=0}^{\infty} \varphi(a^n)\lambda^n$ for $|\lambda| < 1/\|a\|$. Moreover, f is holomorphic for $|\lambda| < 1/r(a)$, so this power series converges to $f(\lambda)$ for $|\lambda| < 1/r(a)$. Hence for $R > r(a)$ we have

$$\varphi(a^n) = \frac{1}{2\pi i} \int_{|\lambda|=1/R} \frac{f(\lambda)}{\lambda^{n+1}}\, d\lambda$$

and we obtain the estimate

$$|\varphi(a^n)| \leq \frac{1}{2\pi} \cdot \frac{2\pi}{R} \cdot R^{n+1} \cdot \sup_{|\lambda|=1/R} |\varphi(S(\lambda))| \leq R^n M(R)$$

where $M(R) = \sup_{|\lambda|=1/R} \|S(\lambda)\|$, which is finite by the continuity of S on the compact set $\{\lambda \in \mathbb{C}\colon |\lambda| = 1/R\}$. Since $S(\lambda) \neq 0$ for any λ in the domain of S, we have $M(R) > 0$. Hence

$$\limsup_{n\geq 1} \|a^n\|^{1/n} \leq \limsup_{n\geq 1} RM(R)^{1/n} = R$$

whenever $R > r(a)$. We conclude that

$$r(a) \leq \inf_{n\geq 1} \|a^n\|^{1/n} \leq \liminf_{n\geq 1} \|a^n\|^{1/n} \leq \limsup_{n\geq 1} \|a^n\|^{1/n} \leq r(a)$$

and the result follows. $\square$

1.3.13 Corollary. *If A is a unital Banach algebra and B is a closed unital subalgebra of A then $r_A(b) = r_B(b)$ for all $b \in B$.*

Proof. The norm of an element of B is the same whether we measure it in B or in A. By the spectral radius formula, $r_A(b) = \lim_{n\geq 1} \|b^n\|^{1/n} = r_B(b)$. $\square$

While the spectral radius of an element of a Banach algebra does not depend if we compute it in a subalgebra, the spectrum itself can change. We explore this in the next few results.

Suppose that A is a unital Banach algebra and B is a unital Banach subalgebra of A. If an element of b is invertible in B, then it is invertible in A; so $\operatorname{Inv} B \subseteq B \cap \operatorname{Inv} A$. However, this inclusion may be strict, as the following example shows.

1.3.14 Example. Recall that $A(\overline{\mathbb{D}})$ is the disc algebra of continuous functions $\overline{\mathbb{D}} \to \mathbb{C}$ which are holomorphic on $\mathbb{D}$. Note that, by the maximum modulus principle, $\sup_{z\in\overline{\mathbb{D}}} |f(z)| = \sup_{\theta\in[0,2\pi)} |f(e^{i\theta})|$. Hence $\|f\| = \|f|_{\mathbb{T}}\|$, and the map $A(\overline{\mathbb{D}}) \to C(\mathbb{T})$, $f \mapsto f|_{\mathbb{T}}$ is a unital isometric isomorphism. So we may identify $A(\overline{\mathbb{D}})$ with $A(\mathbb{T}) = \{f|_{\mathbb{T}} \colon f \in A(\overline{\mathbb{D}})\}$, which is a closed unital subalgebra of $C(\mathbb{T})$.

Consider the function $f(z) = z$ for $z \in \overline{\mathbb{D}}$. This is not invertible in $A(\overline{\mathbb{D}})$ since $f(0) = 0$. Hence $f|_{\mathbb{T}}$ is not invertible in $A(\mathbb{T})$. However, f is invertible in $C(\mathbb{T})$ with inverse $g \colon e^{i\theta} \mapsto e^{-i\theta}$. So $\operatorname{Inv} A(\mathbb{T}) \subsetneq A(\mathbb{T}) \cap \operatorname{Inv} C(\mathbb{T})$.

1.3.15 Definition. If A is a unital Banach algebra then a subalgebra $B \subseteq A$ with $1 \in B$ is said to be *inverse-closed* if $\operatorname{Inv} B = B \cap \operatorname{Inv} A$; that is, if every $b \in B$ which is invertible in A also has $b^{-1} \in B$.

Clearly, if B is an inverse-closed unital subalgebra of A then $\sigma_B(b) = \sigma_A(b)$ for all $b \in B$.

If K is a non-empty compact subset of $\mathbb{C}$ then exactly one of the connected components of $\mathbb{C} \setminus K$ is unbounded. The bounded components of $\mathbb{C} \setminus K$ are called the *holes* of K. If A is a unital Banach algebra and $a \in A$, let us write

$$R_A(a) = \mathbb{C} \setminus \sigma_A(a) = \{\lambda \in \mathbb{C} \colon \lambda - a \in \operatorname{Inv} A\}.$$

This is sometimes called the *resolvent set* of a. Note that the bounded connected components of $R_A(a)$ are precisely the holes of $\sigma_A(a)$.

1.3.16 Theorem. *Let B be a closed subalgebra of a unital Banach algebra A with $1 \in B$. If $b \in B$ then $\sigma_B(b)$ is the union of $\sigma_A(b)$ with zero or more of the holes of $\sigma_A(b)$. In particular, if $\sigma_A(b)$ has no holes then $\sigma_B(b) = \sigma_A(b)$.*

Proof. Since $\operatorname{Inv} B \subseteq \operatorname{Inv} A$ we have $R_B(b) \subseteq R_A(b)$, and so $\sigma_A(b) \subseteq \sigma_B(b)$, for each $b \in B$. We claim that $R_B(b)$ is a relatively clopen subset of $R_A(b)$. Since $\sigma_B(b)$ is closed by Theorem 1.3.5, $R_B(b)$ is open. The map

$$i\colon R_A(b) \to A, \quad \lambda \mapsto (\lambda - b)^{-1}$$

is continuous by Corollary 1.2.9, and

$$R_B(b) = \{\lambda \in R_A(b)\colon i(\lambda) = (\lambda - b)^{-1} \in B\} = i^{-1}(B).$$

Since B is closed, $R_B(b)$ is closed.

If G is a connected component of $R_A(b)$ then $G \cap R_B(b)$ is either $\emptyset$ or G. For otherwise, since $R_B(b)$ is clopen, $G \cap R_B(b)$ and $G \setminus R_B(b)$ would be proper clopen subsets of the connected set G, which is impossible. If G is the unbounded component of $R_A(b)$ then, since $\sigma_B(b)$ is bounded, we must have $G \cap \sigma_B(b) = \emptyset$. The bounded components of $R_A(b)$ are precisely holes of $\sigma_A(b)$. Hence

$$\sigma_B(b) = \sigma_A(b) \cup \bigcup \{G \text{ a hole of } \sigma_A(b)\colon G \cap \sigma_B(b) \neq \emptyset\}.$$

If $\sigma_A(b)$ has no holes then this reduces to $\sigma_B(b) = \sigma_A(b)$. □

1.3.17 Definition. Let A be a Banach algebra. If $S \subseteq A$ then the *commutant* of S in A is

$$S' = \{a \in A\colon ab = ba \text{ for all } b \in S\}.$$

The *bicommutant* of S in A is $S'' = (S')'$.

A set $S \subseteq A$ is *commutative* if $ab = ba$ for all $a, b \in S$. Hence S is commutative if and only if $S \subseteq S'$.

1.3.18 Lemma. *Let A be a Banach algebra. If $T \subseteq S \subseteq A$, then $T' \supseteq S'$. Moreover, $S \subseteq S''$ and $S' = S'''$.*

Proof. Exercise. □

1.3.19 Proposition. *Let A be a unital Banach algebra and let $S \subseteq A$.*

(i). S' is a closed, inverse-closed unital subalgebra of A.

(ii). If S is commutative then so is $B = S''$, and $\sigma_B(b) = \sigma_A(b)$ for all $b \in B$.

Proof. (i) Since multiplication is continuous on A, it is easy to see that the commutant S' is closed. Clearly $1 \in S'$, and using the linearity of multiplication shows that S' is a vector subspace of A, and it is a subalgebra by associativity. If $b \in S' \cap \operatorname{Inv} A$ then $bc = cb$ for all $c \in S$, so $cb^{-1} = b^{-1}c$ for all $c \in S$, so $b^{-1} \in S'$ and S' is inverse-closed.

(ii) We have $S \subseteq S'$, so $S' \supseteq S''$ and $S'' \subseteq S'''$ by Lemma 1.3.18. Hence $B = S''$ is commutative. Moreover, B is an inverse-closed subalgebra of A by (i), so $\sigma_B(b) = \sigma_A(b)$ for all $b \in B$. □

1.3.20 Definition. Let A be a Banach algebra without an identity element. The *unitisation* of A is the Banach algebra $\widetilde{A}$ whose underlying vector space is $A \oplus \mathbb{C}$ with the product $(a, \lambda)(b, \mu) = (ab+\lambda b+\mu a, \lambda\mu)$ and norm $\|(a, \lambda)\| = \|a\| + |\lambda|$ for $a, b \in A$ and $\lambda, \mu \in \mathbb{C}$. Note that $\widetilde{A}$ is then a unital Banach algebra containing A (or, more precisely, $A \times \{0\}$).

1.3.21 Definition. If A has no identity element and $a \in A$, then we define $\sigma_A(a) = \sigma_{\widetilde{A}}(a)$. In this case we have $0 \in \sigma(A)$ for all $a \in A$.

1.3.22 Remark. With this definition, many of the important theorems above apply to non-unital Banach algebras, simply by considering $\widetilde{A}$ instead of A. In particular, it is easy to check that non-unital versions of Theorems 1.3.5, 1.3.8 and 1.3.12 hold.

1.4 Quotients of Banach spaces

Recall that if K is a subspace of a complex vector space X, then the quotient vector space X/K is given by

$X/K = \{x + K : x \in X\}$
with scalar multiplication $\lambda(x + K) = \lambda x + K, \quad \lambda \in \mathbb{C},\ x \in X$
and vector addition $(x + K) + (y + K) = (x + y) + K, \quad x, y \in X,\ \lambda \in \mathbb{C}$.

The zero vector in X/K is $0 + K = K$.

1.4.1 Definition. If K is a closed subspace of a Banach space X then the *quotient Banach space* X/K is the vector space X/K equipped with the *quotient norm*, defined by

$$\|x + K\| = \inf_{k\in K} \|x + k\|.$$

1.4.2 Proposition. *Let X be a Banach space and let K be a closed vector subspace of X. The quotient norm is a norm on the vector space X/K, with respect to which X/K is complete. Hence the quotient Banach space X/K is a Banach space.*

Proof. To see that the quotient norm is a norm, observe that:

- $\|x + K\| \geq 0$ with equality if and only if $\inf\limits_{k \in K} \|x + k\| = 0$, which is equivalent to x being in the closure of K; since K is closed, this means that $x \in K$ so $x + K = K$, the zero vector of X/K.
- If $\lambda \in \mathbb{C}$ with $\lambda \neq 0$ then

$$\begin{aligned}\|\lambda(x+K)\| &= \inf_{k \in K} \|\lambda x + k\| = |\lambda| \inf_{k \in K} \|x + \lambda^{-1} k\| \\ &= |\lambda| \inf_{k' \in K} \|x + k\| = |\lambda| \, \|x + K\|.\end{aligned}$$

- The triangle inequality holds since $K = \{s + t \colon s, t \in K\}$ and so

$$\begin{aligned}\|(x+K) + (y+K)\| = \|x + y + K\| &= \inf_{k \in K} \|x + y + k\| \\ &= \inf_{s,t \in K} \|x + y + s + t\| \\ &\leq \inf_{s \in K} \|x + s\| + \inf_{t \in K} \|y + t\| \\ &= \|x + K\| + \|y + K\|.\end{aligned}$$

It remains to show that X/K is complete in the quotient norm. For any $x \in X$, it is not hard to see that:

(i) if $\varepsilon > 0$ then there exists $k \in K$ such that $\|x + k\| < \|x + K\| + \varepsilon$; and

(ii) $\|x + K\| \leq \|x\|$ (since $0 \in K$).

Let $x_j \in X$ with $\sum_{j=1}^{\infty} \|x_j + K\| < \infty$. From observation (i), it follows that there exist $k_j \in K$ with $\sum_{j=1}^{\infty} \|x_j + k_j\| < \infty$, so by [FA 1.7.8] the series $\sum_{j=1}^{\infty} x_j + k_j$ converges in X, say to $s \in X$. By observation (ii),

$$\Big\| s + K - \Big(\sum_{j=1}^{n} x_j + K \Big) \Big\| = \Big\| \Big(s - \sum_{j=1}^{n} x_j + k_j \Big) + K \Big\| \leq \Big\| s - \sum_{j=1}^{n} x_j + k_j \Big\| \to 0$$

as $n \to \infty$, so $\sum_{j=1}^{\infty} x_j + k_j$ converges to $s + K$. This shows that every absolutely convergent series in X/K is convergent with respect to the quotient norm, so X/K is complete by [FA 1.7.8]. $\square$

1.5 Ideals, quotients and homomorphisms of Banach algebras

1.5.1 Definition. An *ideal* of a Banach algebra A is a vector subspace I of A such that for all $x \in I$ and $a \in A$ we have $ax \in I$ and $xa \in I$.

1.5.2 Definition. Let I be a closed ideal of a Banach algebra A. The *quotient Banach algebra* A/I is the quotient Banach space A/I equipped with the product $(a+I)(b+I) = ab+I$ for $a, b \in I$.

1.5.3 Theorem. *If I is a closed ideal of a Banach algebra A then A/I is a Banach algebra. If A is abelian then so is A/I. If A is unital then so is A/I, and $1_{A/I} = 1_A + I$.*

Proof. We saw in Proposition 1.4.2 that A/I is a Banach space. Just as for quotient rings, the product is well-defined, since if $a_1 + I = a_2 + I$ and $b_1 + I = b_2 + I$ then $a_1 - a_2 \in I$ and $b_1 - b_2 \in I$, so $a_1(b_1 - b_2) + (a_1 - a_2)b_2 \in I$ and so

$$(a_1b_1 + I) - (a_2b_2 + I) = a_1b_1 - a_2b_2 + I = a_1(b_1 - b_2) + (a_1 - a_2)b_2 + I = I,$$

hence $a_1b_1 + I = a_2b_2 + I$. It is easy to see that this product is linear in each variable.

Let $a, b \in A$. We have

$$\begin{aligned}
\|a+I\|\,\|b+I\| &= \inf_{y,z\in I} \|a+y\|\,\|b+z\| \\
&\geq \inf_{y,z\in I} \|(a+y)(b+z)\| \text{ (by 1.1.1(iii) in } A) \\
&= \inf_{y,z\in I} \|ab + (az + yb + yz)\| \\
&\geq \inf_{x\in I} \|ab + x\| \text{ (since } az + by + yz \in I \text{ for all } y, z \in I) \\
&= \|ab + I\| = \|(a+I)(b+I)\|.
\end{aligned}$$

Hence the inequality 1.1.1(iii) holds in A/I, and we have shown that A/I is a Banach algebra.

If A is abelian then $(a+I)(b+I) = ab+I = ba+I = (b+I)(a+I)$ for all $a, b \in A$, so A/I is abelian. The proof of the final statement about units is left as an exercise. □

1.5.4 Definition. A *proper ideal* of a Banach algebra A an ideal of A which is not equal to A. A *maximal ideal* of A is a proper ideal such which is not contained in any strictly larger proper ideal of A.

1.5.5 Lemma. *Let A be a unital Banach algebra. If I is an ideal of A, then I is a proper ideal if and only if $I \cap \operatorname{Inv} A = \emptyset$.*

Proof. We have $1 \in \operatorname{Inv} A$, so if $I \cap \operatorname{Inv} A = \emptyset$ then $1 \notin I$, so $I \neq A$ and I is a proper ideal. Conversely, if $I \cap \operatorname{Inv} A \neq \emptyset$, let $b \in I \cap \operatorname{Inv} A$. If $a \in A$ then $a = (ab^{-1})b \in I$, since I is an ideal and $b \in I$. So $I = A$. □

1.5.6 Theorem. *Let A be a unital Banach algebra.*

(i). If I is a proper ideal of A then the closure $\overline{I}$ is also a proper ideal of A.

(ii). Any maximal ideal of A is closed.

Proof. (i) The closure of a vector subspace of A is again a vector subspace. If $a \in A$ and x_n is a sequence in I converging to $x \in \overline{I}$ then $ax_n \to ax$ and $x_n a \to xa$ as $n \to \infty$. Since each ax_n and $x_n a$ is in I, this shows that ax and xa are in $\overline{I}$, which is therefore an ideal of A.

Since I is a proper ideal we have $I \cap \operatorname{Inv} A = \emptyset$ by Lemma 1.5.5. Since $\operatorname{Inv} A$ is open by Corollary 1.2.8, this shows that $\overline{I} \cap \operatorname{Inv} A = \emptyset$ so $\overline{I} \neq A$.

(ii) Let M be a maximal ideal. Since $M \subseteq \overline{M}$ and $\overline{M}$ is a proper ideal by (i), we must have $M = \overline{M}$, so M is closed. □

1.5.7 Remarks. (i). Since we know a few results about ideals of rings, we would like to apply these to ideals of Banach algebras. Any unital Banach algebra A may be viewed as a unital ring R by ignoring the norm and scalar multiplication. However, there is a difference in the definitions: ideals of R are not required to be linear subspaces (since R has no linear structure) whereas ideals of A are. However, the two definitions turn out to be equivalent if A is unital. Indeed, an ideal of the Banach algebra A is clearly an ideal of the ring R. Conversely, if I is an ideal of the ring R then since $\lambda 1 \in R$ for $\lambda \in \mathbb{C}$ we have $\lambda x = \lambda 1 \cdot x \in I$ for all $x \in I$, so I *is* a vector subspace of A with the ideal property. So I is an ideal of A.

(ii). By [FA 2.16], any proper ideal of a unital Banach algebra A is contained in a maximal ideal of A.

1.5.8 Definition. Let A and B be Banach algebras. A *homomorphism* from A to B is a linear map $\theta\colon A \to B$ which is multiplicative in the sense that $\theta(ab) = \theta(a)\theta(b)$ for all $a, b \in A$.

The *kernel* of such a homomorphism θ is the set

$$\ker \theta = \{a \in A\colon \theta(a) = 0\}.$$

If A and B are unital Banach algebras, we say that a homomorphism $\theta\colon A \to B$ is *unital* if $\theta(1_A) = 1_B$.

A bijective homomorphism $\theta\colon A \to B$ is an *isomorphism.* If such an isomorphism exists then the Banach algebras A and B are *isomorphic.* It is easy to see that if θ is an isomorphism then so is θ^{-1}.

1.5.9 Remark. If $\theta\colon A \to B$ is a non-zero homomorphism of Banach algebras then $\ker\theta$ is an ideal of A, which is proper unless $\theta = 0$. If θ is continuous then $\ker\theta$ is a closed ideal of A.

1.5.10 Remark. We usually say that two objects are isomorphic if they have the same structure; that is, if they are the same "up to relabelling". However, if two Banach algebras A and B are isomorphic then this tells us that they have the same structure as algebras, but not necessarily as Banach algebras, since the norms may not be related.

The strongest notion of "the same Banach algebra up to relabelling" is *isometric isomorphism.* Two Banach algebras A and B are isometrically isomorphic if there is an isomorphism $\theta\colon A \to B$ which is also an isometry, meaning that $\|\theta(a)\| = \|a\|$ for all $a \in A$ (compare with [FA 1.3.8]).

1.5.11 Examples. (i). If A is a non-unital Banach algebra then the map $\theta\colon A \to \widetilde{A}$, $a \mapsto (a, 0)$ from Definition 1.3.20 is an isometric homomorphism. Hence $\theta(A)$ is a Banach subalgebra of $\widetilde{A}$ which is isometrically isomorphic to A.

(ii). The map $A(\overline{\mathbb{D}}) \to A(\mathbb{T})$, $f \mapsto f|_{\mathbb{T}}$ from Example 1.3.14 is a unital isometric isomorphism.

1.5.12 Proposition. *Let A and B be unital Banach algebras and let $\theta\colon A \to B$ be a unital homomorphism.*

(i). $\theta(\operatorname{Inv} A) \subseteq \operatorname{Inv} B$, and $\theta(a)^{-1} = \theta(a^{-1})$ for $a \in \operatorname{Inv} A$.

(ii). For all $a \in A$ we have $\sigma_A(a) \supseteq \sigma_B(\theta(a))$.

(iii). If θ is an isomorphism then $\sigma_A(a) = \sigma_B(\theta(a))$ for all $a \in A$.

Proof. (i) If $a \in \operatorname{Inv} A$ then $\theta(a)\theta(a^{-1}) = \theta(aa^{-1}) = \theta(1) = 1$ and $\theta(a^{-1})\theta(a) = \theta(a^{-1}a) = \theta(1) = 1$, so $\theta(a)$ is invertible in B, with inverse $\theta(a^{-1})$.

(ii) If $\lambda \in \sigma_B(\theta(a))$ then $\lambda - \theta(a) = \theta(\lambda - a) \notin \operatorname{Inv} B$ so $\lambda - a \notin \operatorname{Inv} A$ by (i). Hence $\lambda \in \sigma_A(a)$.

(iii) Since θ^{-1} is a homomorphism, by (ii) we have

$$\sigma_A(a) = \sigma_A(\theta^{-1}(\theta(a))) \subseteq \sigma_B(\theta(a)) \subseteq \sigma_A(a),$$

and we have equality. $\square$