UNIVERSITY OF DUBLIN

XMA442C

TRINITY COLLEGE

Faculty of Engineering, Mathematics and Science

school of mathematics

SS Mathematics **Trinity Term 2010**
SS Two Subject Moderatorship

MA442C — Banach Algebras

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Credit will be given for the best 3 questions answered.

All questions have equal weight.

Note: this sample exam is intended to give you an idea of the general format of the paper, but not of the topics it will cover.

1. Let A be a unital Banach algebra, and let $a \in A$.

   (a) [3 marks] What does it mean to say that a is *invertible*?

   (b) [6 marks] Suppose that $a_1, a_2, \ldots, a_n$ are commuting elements of A such that

   $$a = a_1 a_2 \ldots a_n.$$

   Prove that if a is invertible, then each of $a_1, \ldots, a_n$ must be invertible.

   (c) [7 marks] Define the spectrum of a, and prove that the spectrum of a is non-empty and compact. You may assume any results from the course concerning properties of the invertible elements of A, and the map $a \mapsto a^{-1}$, provided you state them clearly.

   (d) [8 marks] By considering the Banach algebras $C(X)$ where X is a compact topological space, or otherwise:

   (i). Prove that if K is any non-empty compact subset of $\mathbb{C}$, then there is a Banach algebra A and an element of A whose spectrum is K.

   (ii). Find a Banach algebra A such that no element of A has spectrum $\{0, 1\}$.

2. Let A be a unital abelian Banach algebra.

   (a) [5 marks] Define the *character space* and the *Gelfand representation* of A.

   (b) [6 marks] Prove that the Gelfand representation of A is a unital norm-decreasing homomorphism.

   (c) [6 marks] Give an example to show that the Gelfand representation need not be injective.

   (d) [7 marks] State the Gelfand-Naimark theorem, and use it to prove that if A is a unital abelian C*-algebra and $a, b \in A$ with $a = a^*$ and $b = b^*$, and $a^3 = b^3$, then $a = b$.

3. Let A be a unital C*-algebra.

   (a) [6 marks] Define the terms *normal*, *hermitian*, and *unitary* as applied to elements of A.

   (b) [5 marks] Carefully prove that if $a \in A$ is normal, then the smallest unital C*-subalgebra of A containing a is abelian.

   (c) [7 marks] Prove that an element $u \in A$ is unitary if and only if u is normal and $\sigma(u) \subseteq \mathbb{T}$.

   (d) [6 marks] If $a \in A$ and a is normal, explain how the continuous functional calculus is used to define $\exp(a)$, where $\exp\colon \mathbb{C} \to \mathbb{C}$, $z \mapsto e^z$ is the exponential function, and explain why $\exp(a)$ is invertible.

4. Let H be a be Hilbert space.

   (a) [5 marks] Define the Hilbert space $H \oplus H$.
   [Don't forget to define the norm on this space].

   (b) [7 marks] Given $S, T \in \mathcal{B}(H)$, define the operator $S \oplus T$ on $H \oplus H$. Give a formula explaining the relationship between

   $$\|S \oplus T\|, \qquad \|S\| \quad \text{and} \quad \|T\|,$$

   and prove that it is correct.

   (c) [12 marks] If A is a C*-subalgebra of $B(H)$, let $B_1 = \{S \oplus S\colon S \in A\}$ and let $B_2 = \{S \oplus T\colon S \in A\}$. Either prove or give a counterexample to each of the following statements:

   (i). B_1 and B_2 are C*-subalgebras of $B(H \oplus H)$

   (ii). B_1 is *-isomorphic to A

   (iii). B_2 is *-isomorphic to A

5. [24 marks] State the Banach–Alaoglu theorem and give an outline of its proof. Then explain precisely what is meant by the following statement, and prove it:

   if A is a unital abelian Banach algebra, then $\Omega(A)$ is a compact and Hausdorff.

   You should aim to give the reader both a precise statement of these results, and a good idea of why they are true. The amount of detail to include is up to you, and you may choose to omit some technical details in proofs of auxiliary results. However, you should include statements and definitions for the most important results, notation and terminology that you use.