Some useful facts about vector spaces over fields

Let F be a field.

Definition. A vector space over F is a set V with two maps, $+\colon V\times V\to V$ and $\cdot\colon F\times V\to V$ so that:

(a) $(V,+)$ is an abelian group

(b) for all $\lambda,\mu\in F$ and all $v,w\in V$ we have

$$\lambda\cdot(v+w)=\lambda\cdot v+\lambda\cdot w,\quad (\lambda+\mu)\cdot v=\lambda\cdot v+\lambda\cdot w,\quad \lambda\cdot(\mu\cdot v)=(\lambda\mu)\cdot v,\quad \text{and}\quad 1_F\cdot v=v.$$

We often call the elements of V vectors, the elements of F scalars, and we call the operation $+$ vector addition, and $\cdot$ scalar multiplication. We usually also write the identity element of $(V,+)$ as the zero vector, 0, and the inverse of $v\in V$ in $(V,+)$ is written as $-v$.

Definition. Let $W=\{w_1,w_2,\dots,w_n\}$ be a finite subset of V.

(a) W is linearly independent (over F) if the only $\lambda_1,\dots,\lambda_n\in F$ with $\lambda_1w_1+\cdots+\lambda_nw_n=0$ are $\lambda_1=0$, $\lambda_2=0,\dots\lambda_n=0$

(b) W spans V (over F) if every $v\in V$ can be written as $v=\lambda_1w_1+\cdots+\lambda_nw_n$ for some $\lambda_1,\dots,\lambda_n\in F$.

(c) W is a basis for V (over F) if it is both linearly independent, and spans V.

Fact 1. (a) V has no basis if and only if, for every $n\ge 1$ there is a linearly independent subset of V which has size n.

(b) If V has a basis, then any two bases of V have the same size.

Definition.

$$\dim(V)=\dim_F(V)=\begin{cases}\infty & \text{if } V \text{ has no basis}\\ |W| & \text{if } V \text{ has a basis } W\end{cases}$$

This is well-defined by the previous fact. We call $\dim(V)$ the dimension of V (over F). If $\dim(V)\ne\infty$ we say V is finite-dimensional (over F), or write $\dim(V)<\infty$.

Fact 2. Suppose that V is finite-dimensional. If W is a linearly independent subset of V, then $|W|\le\dim(V)$. Moreover,

(a) $|W|=\dim(V)\iff W$ is a basis for V, and

(b) if $|W|<\dim(V)$ then $W\subseteq B$ for some basis B of V.

Definition. A (vector) subspace of V is a subset U which is also a vector space over F (with the same (actually, the restrictions of) vector addition and scalar addition on V).

Fact 3. If U is a subspace of V, then $\dim(U)\le\dim(V)$. If $\dim(V)<\infty$ then

$$U=V\iff\dim(U)=\dim(V).$$

(If $\dim(V)=\infty$ then this isn't true).

These facts should all be familiar to you for vector spaces over $\mathbb{R}$. The proofs are practically identical for vector spaces over F, where F is any field.