UNIVERSITY OF DUBLIN

XMA2215

TRINITY COLLEGE

Faculty of Engineering, Mathematics
and Science

school of mathematics

SF Mathematics **Trinity Term 2011**
SF Two Subject Moderatorship

MA2215 — Fields, rings and modules

2 hours

Dr. R. Levene

Credit will be given for the best 4 questions answered.

All questions have equal weight.

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.

This sample is intended to give you some idea of the exam format, but it is *not* a guide to the topics from the course that will appear.

1. (a) [7 marks] Let R be a ring and let $x_1, x_2, y_1, y_2 \in R$. State the definition of an ideal of R, and show that if I is an ideal of R with

$$I + x_1 = I + x_2 \quad \text{and} \quad I + y_1 = I + y_2,$$

then

$$I + (x_1 + y_1) = I + (x_2 + y_2) \quad \text{and} \quad I + x_1 y_1 = I + x_2 y_2.$$

(b) [6 marks] State and prove the first isomorphism theorem for rings.

(c) [7 marks] Let R be a ring with no zero-divisors which contains at least two elements. Consider the subring of $M_2(R)$ given by

$$S = \left\{ \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} : a, b, c \in R \right\}.$$

Prove that S is not isomorphic to R, and find an ideal $I \lhd S$ so that S/I is isomorphic to R.

2. (a) [7 marks] For each of the following properties, EITHER give an example of a ring with those properties, OR explain why no such ring exists. In both cases, briefly explain why your answer is correct.

(i). A non-commutative ring with a commutative subring.

(ii). An integral domain which is not isomorphic to a subring of a field.

(iii). A unique factorisation domain containing two irreducible elements p, q so that $p | q$ and $p \neq q$.

(b) [6 marks] Let R be an integral domain. Recall that for $a, b \in R$, we write $a \| b$ to mean that a and b are associates in R.

Explain what it means to say that a and b are associates in R. Then show that if $x_1 \| x_2$ and $y_1 \| y_2$, then $x_1 x_2 \| y_1 y_2$, but we need not have $(x_1 + x_2) \| (y_1 + y_2)$.

(c) [7 marks] Explain how you know that the ring $\mathbb{Z}[i]$ of Gaussian integers is a unique factorisation domain. Then write $11 - 3i$ as a product of irreducible elements of $\mathbb{Z}[i]$.

3. Consider $\mathbb{Z}_5[x]$, the ring of polynomials with coefficients in $\mathbb{Z}_5$, the integers modulo 5.

   (a) [6 marks] Compute the gcds of x^3+4 and x^5+3x+1 in $\mathbb{Z}_5[x]$.

   (b) [7 marks] State the definition of a maximal ideal of R, and show that an ideal I of $\mathbb{Z}_5[x]$ is a maximal ideal if and only if $I = \langle f \rangle$ where f is an irreducible polynomial in $\mathbb{Z}_5[x]$.

   (c) [7 marks] Prove that $\mathbb{Z}_5[x]/\langle x \rangle \approx \mathbb{Z}_5$, and find an ideal I of $\mathbb{Z}_5[x]$ so that $\mathbb{Z}_5[x]/I$ is a field but is not isomorphic to $\mathbb{Z}_5$.

4. (a) [7 marks] State and prove the Tower Law for field extensions.
   [You can restrict attention in your proof to extensions of finite degree.]

   (b) [6 marks] Suppose that K is a field extension of a field F with $F \subseteq K$. Prove that if $[K : F] = 14$, then there exist α, β in K so that $K = F(\alpha, \beta)$.

   (c) [7 marks] Suppose that α and β are non-zero complex numbers which are algebraic over $\mathbb{Q}$. Show that if $\theta\colon \mathbb{Q}(\alpha) \to \mathbb{Q}(\beta)$ is a ring homomorphism so that $\theta(\alpha) = \beta$, then θ is an isomorphism. Must we have $\alpha = \beta$?

5. [20 marks] In the context of ruler-and-compass constructions, state the definition of a *constructible* point in the plane. Determine which of the following points are constructible:

$$(\tfrac{2}{3}, 0), \qquad (2^{1/3}, 0)$$

Prove any results about ruler-and-compass constructions that you need.
[You may use general results about rings and fields from the course without proof.]