Problem session solutions

1. Write the following permutations as a product of disjoint cycles:

   (a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 4 & 3 \end{pmatrix}$

   (b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix}$

   (c) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 2 & 3 & 5 & 6 & 7 & 1 & 4 \end{pmatrix}$

   Solution $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 4 & 3 \end{pmatrix} = (1\ 5\ 3\ 2)$, $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix} = (1\ 5)(2\ 4)$ and $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 2 & 3 & 5 & 6 & 7 & 1 & 4 \end{pmatrix} = (1\ 8\ 4\ 5\ 6\ 7)$.

2. Write the following permutations as a product of disjoint cycles, and in 2-row notation:

   (a) $(3\ 2\ 7\ 4)(7)$, as an element of S_8

   (b) $(1\ 2\ 3\ 5)(4\ 5\ 3)$, as an element of S_5

   (c) $(1\ 2)(3\ 5\ 1)$, as an element of S_5

   (d) $(1\ 6)(1\ 5)(1\ 4)(1\ 3)(1\ 2)$, as an element of S_7

   Solution (a) $(3\ 2\ 7\ 4)(7) = (3\ 2\ 7\ 4)$, since (7) is the identity permutation, and $(3\ 2\ 7\ 4) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 7 & 2 & 3 & 5 & 6 & 4 & 8 \end{pmatrix}$.

   (b) $(1\ 2\ 3\ 5)(4\ 5\ 3) = (1\ 2\ 3\ 4)(5) = (1\ 2\ 3\ 4) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 1 & 5 \end{pmatrix}$.

   (c) $(1\ 2)(3\ 5\ 1) = (1\ 3\ 5\ 2) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 5 & 4 & 2 \end{pmatrix}$.

   (d) $(1\ 6)(1\ 5)(1\ 4)(1\ 3)(1\ 2) = (1\ 2\ 3\ 4\ 5\ 6) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 5 & 6 & 1 & 7 \end{pmatrix}$.

3. Recall that if $\alpha \in S_n$ and $k \in \mathbb{N}$ then $\alpha^k = \underbrace{\alpha \circ \cdots \circ \alpha}_{k \text{ times}}$. Let us call a cycle non-trivial if it is not the identity mapping.

   (a) Find a cycle α such that α^3 is the product of 3 non-trivial disjoint cycles.

   (b) Find a cycle α such that α^{12} is the product of 4 non-trivial disjoint cycles.

   Solution (a) For example, if $\alpha = (1\ 2\ 3\ 4\ 5\ 6) \in S_6$, then $\alpha^3 = (1\ 4)(2\ 5)(3\ 6)$.
   (b) For example, if $\alpha = (1\ 2\ 3\ \ldots\ 19\ 20) \in S_{20}$, then

   $$\alpha^{12} = (1\ 13\ 5\ 17\ 9)(2\ 14\ 6\ 18\ 10)(3\ 15\ 7\ 19\ 11)(4\ 16\ 8\ 20\ 12).$$

4. If S is any set, a permutation $\alpha \in \mathrm{Sym}(S)$ is said to be a cycle if there are finitely many elements $a_1, \ldots, a_k \in S$ with $\alpha(a_j) = a_{j+1}$ for $0 \leq j < k$ and $\alpha(a_k) = a_1$, and $\alpha(x) = x$ if $x \notin \{a_1, \ldots, a_k\}$.

   Consider the mapping $\alpha \colon \mathbb{Z} \to \mathbb{Z}$, $n \mapsto n + 1$. Explain why $\alpha \in \mathrm{Sym}(\mathbb{Z})$ and show that α is not equal to a composition of a finite number of cycles.

Solution We have $\alpha(n) = \alpha(m) \iff n+1 = m+1 \iff n = m$, so α is one-to-one, and if $m \in \mathbb{Z}$ then $m-1 \in \mathbb{Z}$ and $\alpha(m-1) = m$, so α is onto. Hence $\alpha \in \mathrm{Sym}(\mathbb{Z})$.

Suppose that α is a composition of finitely many cycles, say

$$\alpha = \beta_1 \circ \beta_2 \circ \cdots \circ \beta_n$$

for some $n \geq 1$, where each β_i is a cycle.
[We will show that this leads to a contradiction, so that this cannot be possible.]

Consider a fixed i with $1 \leq i \leq n$. Since $\mathbb{Z}$ is infinite and β_i only moves finitely many elements of $\mathbb{Z}$, there is an integer $m_i \in \mathbb{Z}$ such that $\beta_i(x) = x$ for all $x \in \mathbb{Z}$ with $x \geq m_i$.

Now let $m = \max\{m_1, \ldots, m_n\}$. We have $\beta_i(m) = m$ for each i with $1 \leq i \leq n$, so

$$m + 1 = \alpha(m) = (\beta_1 \circ \beta_2 \circ \cdots \circ \beta_n)(m) = m,$$

which is a contradiction. So α cannot be a composition of finitely many cycles.