Mathematics 1214: Introduction to Group Theory

Homework exercise sheet 6

Due 12:50pm, Friday 12th March 2010

1. Find $M_{(T)}$ for the following sets $T \subseteq P$. If $M_{(T)}$ has finite order, give its Cayley table.
   (a) The circle of radius 1 centred at the origin (this circle is given by $x^2 + y^2 = 1$)
   (b) The ellipse given by $x^2 + 2y^2 = 4$ (you can plot this using the fact that it's an ellipse, finding its x and y intercepts and the symmetry of the equation)
   (c) The filled in square given by $|x| + |y| \leq 1$

Solution (a) We claim that $M_{(T)} = M_{\{0\}} = \{\alpha \in M : \alpha(0) = 0\}$.

If $\alpha \in M_{\{0\}}$ then $\alpha \in M$, so $\alpha = \tau_a \circ \rho_\theta \circ s$ where $s = r$ or $s = \iota_P$, by Theorem 18, and $\alpha(0) = a = 0$, so $\alpha = \rho_\theta$ or $\alpha = \rho_\theta \circ r$. Since $s(T) = T$ and $\rho_\theta(T) = T$ and $M_{(T)}$ is a subgroup of M, we have $\alpha = \rho_\theta \circ s \in M_{(T)}$. Hence $M_{\{0\}} \subseteq M_{(T)}$.

On the other hand, if $\alpha \in M_{(T)}$ then $2 = d(e_1, -e_1) = d(\alpha(e_1), \alpha(-e_1))$ and $\alpha(e_1), \alpha(-e_1) \in T$. Since the only pairs of points in T which are distance 2 from one another are opposite points, $\alpha(e_1)$ and $\alpha(-e_1)$ are opposite points on T. Now $d(\alpha(0), \alpha(\pm e_1)) = d(0, \pm e_1) = 1$ so $\alpha(0)$ lies in the intersection of the circles of radius 1 centred at $\alpha(e_1)$ and $\alpha(-e_1)$. Since these are opposite points on the unit circle, this forces $\alpha(0) = 0$. So $\alpha \in M_{\{0\}}$. Hence $M_{(T)} \subseteq M_{\{0\}}$.

[Alternatively, we could simply "observe" that $M_{(T)}$ is the set of motions not involving translations, but it's nice to be able to write down a rigorous argument.]

Hence $M_{(T)} = M_{\{0\}} = \{\rho_\theta \circ s \colon \theta \in \mathbb{R},\ s \in \{\iota_P, r\}\}$.

Clearly, this group has infinite order (for example, the rotations ρ_θ for θ in the infinite set $[0, 2\pi)$ are all different, so there are infinitely many of them).

(b) By drawing a picture as suggested, we observe that the points $2e_1$ and $-2e_1$ are in T and $d(2e_1, -2e_1) = 4$, which is larger than $d(p, q)$ for any other pair $p, q \in T$. Hence if $\alpha \in M_{(T)}$ then $\alpha(\{2e_1, -2e_1\}) = \{2e_1, -2e_1\}$ and, arguing as in (a), this forces $\alpha(0) = 0$. So $\alpha = \rho_\theta \circ s$ for $s = \iota_P$ or $s = r$. If $s = \iota_P$ then $\alpha(\{2e_1, -2e_1\}) = \{2e_1, -2e_1\}$ forces $\theta = 0$ or $\theta = \pi$, and if $s = r$ then again we must have $\theta = 0$ or $\theta = \pi$. So $M_{(T)} \subseteq \{\iota_P, \rho_\pi, r, \rho_\pi \circ r\}$. It's easy to check (by drawing a picture) that these four motions α all have the property that $\alpha(T) = T$, so $M_{(T)} = \{\iota_P, \rho_\pi, r, \rho_\pi \circ r\}$. Here's the Cayley table:

$\circ$	ι_P	ρ_π	r	$\rho_\pi \circ r$
ι_P	ι_P	ρ_π	r	$\rho_\pi \circ r$
ρ_π	ρ_π	ι_P	$\rho_\pi \circ r$	r
r	r	$\rho_\pi \circ r$	ι_P	ρ_π
$\rho_\pi \circ r$	$\rho_\pi \circ r$	r	ρ_π	ι_P

(c) Again looking at the pairs of points in T at furthest distance from one another, we can argue that $M_{(T)} = M_{(\{\text{vertices of } T\})} = D_4$. So $M_{(T)} = \{\iota_P, \rho, \rho^2, \rho^3, r, r_0, r_1, r_2, r_3\}$ where $\rho = \rho_{\pi/2}$ and $r_j = \rho^j \circ r$ for $j = 0, 1, 2, 3$. The Cayley table is

$\circ$	ι_P	ρ	ρ^2	ρ^3	r_0	r_1	r_2	r_3
ι_P	ι_P	ρ	ρ^2	ρ^3	r_0	r_1	r_2	r_3
ρ	ρ	ρ^2	ρ^3	ι_P	r_1	r_2	r_3	r_0
ρ^2	ρ^2	ρ^3	ι_P	ρ	r_2	r_3	r_0	r_1
ρ^3	ρ^3	ι_P	ρ	ρ^2	r_3	r_0	r_1	r_2
r_0	r_0	r_3	r_2	r_1	ι_P	ρ^3	ρ^2	ρ
r_1	r_1	r_0	r_3	r_2	ρ	ι_P	ρ^3	ρ^2
r_2	r_2	r_1	r_0	r_3	ρ^2	ρ	ι_P	ρ^3
r_3	r_3	r_2	r_1	r_0	ρ^3	ρ^2	ρ	ι_P

2. Let $T \subseteq P$, fix a motion $\alpha \in M$ and let $S = \alpha(T)$. Prove that

$$M_{(S)} = \{\alpha \circ \beta \circ \alpha^{-1} \colon \beta \in M_{(T)}\}.$$

Solution Let $H = \{\alpha \circ \beta \circ \alpha^{-1} \colon \beta \in M_{(T)}\}$. If $\gamma \in M$, then

$$\begin{aligned}
\gamma \in M_{(S)} &\iff \gamma(S) = S \\
&\iff \gamma(\alpha(T)) = \alpha(T) \quad \text{since } S = \alpha(T) \\
&\iff \alpha^{-1}(\gamma(\alpha(T)) = T \quad \text{by Lemma 14} \\
&\iff (\alpha^{-1} \circ \gamma \circ \alpha)(T) = T \quad \text{since } \lambda(\mu(X)) = (\lambda \circ \mu)(X) \\
&\iff \alpha^{-1} \circ \gamma \circ \alpha \in M_{(T)} \quad \text{by the definition of } M_{(T)} \\
&\iff \alpha^{-1} \circ \gamma \circ \alpha = \beta \text{ for some } \beta \in M_{(T)} \quad \text{since this is what “}\in\text{” means} \\
&\iff \gamma = \alpha \circ \beta \circ \alpha^{-1} \text{ for some } \beta \in M_{(T)} \\
&\qquad\qquad \text{by considering } \alpha \circ LHS \circ \alpha^{-1}\text{, and the invertibility of } \alpha \text{ and } \alpha^{-1} \\
&\iff \gamma \in H \quad \text{by the definition of } H.
\end{aligned}$$

Since both H and $M_{(S)}$ are subsets of M, this shows that $H = M_{(S)}$.

3. Which of the following defines an equivalence relation $\sim$ on $P = \{\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \colon x, y \in \mathbb{R}\}$?
[Hint: two do, and two do not]. Prove that your answers are correct.

(a) $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \sim \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) \iff y - b \in \mathbb{Z}$

(b) $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \sim \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) \iff x - a \in \mathbb{Z}$ or $y - b \in \mathbb{Z}$

(c) $p \sim q \iff d(p, 0) = d(q, 0)$ where $d(\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right), \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)) = \sqrt{(x-a)^2 + (y-b)^2}$

(d) $p \sim q \iff p \cdot q = 0$ where $\cdot$ is the vector dot product $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) = xa + yb$.

For those relations that are equivalence relations, compute the equivalence classes and find a complete set of equivalence class representatives.

Solution (a) This is an equivalence relation. Indeed:

$\sim$ **is reflexive** since $\binom{x}{y} \in P \implies y - y = 0 \in \mathbb{Z} \implies \binom{x}{y} \sim \binom{x}{y}$

$\sim$ **is symmetric** since $\binom{x}{y} \sim \binom{a}{b} \iff y - b \in \mathbb{Z} \iff -(y-b) \in \mathbb{Z} \iff b - y \in \mathbb{Z} \iff \binom{a}{b} \sim \binom{x}{y}$

$\sim$ **is transitive** since $\binom{x}{y} \sim \binom{a}{b}$ and $\binom{a}{b} \sim \binom{c}{d} \iff y - b \in \mathbb{Z}$ and $b - d \in \mathbb{Z} \implies y - b + b - d \in \mathbb{Z} \iff y - d \in \mathbb{Z} \iff \binom{x}{y} \sim \binom{c}{d}$.
In summary, $\binom{x}{y} \sim \binom{a}{b}$ and $\binom{a}{b} \sim \binom{c}{d} \implies \binom{x}{y} \sim \binom{c}{d}$, so $\sim$ is transitive.

The equivalence classes are the sets

$$\left[\binom{a}{b}\right] = \left\{\binom{x}{y} \in P \colon y - b \in \mathbb{Z}\right\} = \left\{\binom{x}{y} \in P \colon y = b + n \text{ for some } n \in \mathbb{Z}\right\} = \left\{\binom{x}{b+n} \colon x \in \mathbb{R},\ n \in \mathbb{Z}\right\}.$$

We claim that a complete set of equivalence class representatives is $R = \left\{\binom{0}{y} \colon 0 \le y < 1\right\}$. Indeed, if $\binom{a}{b} \in \mathbb{R}$ then $\binom{a}{b} \sim \binom{0}{\lfloor b \rfloor}$ where $\lfloor b \rfloor$ is the fractional part of b (that is, the number $b + n$ where $n \in \mathbb{Z}$ is the unique integer such that $0 \le b + n < 1$), so R contains at least one element from every equivalence class. Moreover, $0 \le y_1 < y_2 < 1$ with $y_1 \neq y_2$, say with $y_1 < y_2$, then $0 < y_2 - y_1 < 1$, so $\binom{0}{y_1} \not\sim \binom{0}{y_2}$. Hence $\binom{0}{y_1}$ and $\binom{0}{y_2}$ are in different equivalence classes, so R contains precisely one element from each equivalence class.

(b) This is not an equivalence relation. Indeed, we have $\binom{1/2}{0} \sim \binom{1/2}{1}$ and $\binom{1/2}{1} \sim \binom{1/2}{1/3}$, but $\binom{1/2}{0} \not\sim \binom{1/2}{1/3}$. So $\sim$ is not transitive, so it is not an equivalence relation.

(c) This is an equivalence relation. Indeed:

$\sim$ **is reflexive** since $p \in P \implies d(p,0) = d(p,0) \iff p \sim p$

$\sim$ **is symmetric** since $p \sim q \iff d(p,0) = d(q,0) \iff d(q,0) = d(p,0) \iff q \sim p$

$\sim$ **is transitive** since $p \sim q$ and $q \sim r \iff d(p,0) = d(q,0)$ and $d(q,0) = d(r,0) \implies d(p,0) = d(r,0) \iff p \sim r$.

For $t \ge 0$, we have $d(te_1, 0) = t$ (where te_1 is, of course, the vector $\binom{t}{0}$). Hence

$$[te_1] = \{p \in P \colon d(p,0) = t\}$$

is the circle of radius t if $t > 0$ centred at 0, and $[te_1] = \{0\}$ if $t = 0$. Moreover, it is clear from this geometric interpretation that if $t_1 \neq t_2$ then the circles $[t_1 e_1]$ and $[t_2 e_1]$ do not intersect one another. We claim that every equivalence class is of this form. Indeed, for every $p \in P$, if $t = d(p,0)$ then $p \sim te_1$, so $[p] = [te_1]$, which establishes the claim. It follows from this that $\{te_1 \colon t \ge 0\}$ is a complete set of equivalence classes representatives.

(d) This is not an equivalence relation, since, for example, $e_1 \sim e_2$ and $e_2 \sim -e_1$, but $e_1 \not\sim -e_1$. Hence $\sim$ is not transitive, so it is not an equivalence relation.

4. Let H be a permutation group on a non-empty set S, and consider the relation $\sim$ of H-orbit equivalence on S defined by

$$x \sim y \iff \exists\, \alpha \in H \colon \alpha(x) = y.$$

Theorem 20 shows that $\sim$ is an equivalence relation on S. Compute the equivalence classes and find a complete set of equivalence class representatives for $\sim$ if:

(a) $S = \{1, 2, 3, 4\}$, $H = \{(1), (2\ 3)\}$

(b) $S = \{1, 2, 3, 4\}$, $H = A_4$

(c) $S = \{1, 2, \ldots, n\}$, $T \subseteq S$ and $H = G_T$ where $G = S_n$

(d) $S = \{1, 2, \ldots, n\}$, $T \subseteq S$ and $H = G_{(T)}$ where $G = S_n$

Solution (a) By considering $\alpha(x)$ for $x \in S$ and $\alpha \in H$, we see that the equivalence relation is given by $2 \sim 3$, $3 \sim 2$ and $x \sim x$ for all $x \in S$. So the equivalence classes are $\{\{1\}, \{2, 3\}, \{4\}\}$ and a complete set of equivalence classes representatives is $\{1, 2, 4\}$ (another one is $\{1, 3, 4\}$).

(b) Since $(1\ 2)(3\ 4) \in H = A_4$, we have $1 \sim 2$; since $(1\ 3)(2\ 4) \in H$, we have and $1 \sim 3$, since since $(1\ 4)(2\ 3) \in H$ we have $1 \sim 4$. Hence $[1] = \{1, 2, 3, 4\} = [2] = [3] = [4]$, so $\{1\}$ is a complete set of equivalence class representatives. (And each of the three sets $\{2\}$, $\{3\}$ and $\{4\}$ are, as well). [This is a trivial equivalence relation on S: everything is equivalent to everything else.]

(c) If $t \in T$ then $\alpha(t) = t$ for all $\alpha \in G_T$. Hence $[t] = \{t\}$.

On the other hand, if $x, y \notin T$ with $x \neq y$ then the transposition $(x\ y)$ is in $H = G_T$, so $x \sim y$. Hence $[x] \supseteq \{y \colon y \in S,\ y \notin T\} = S \backslash T$. [This is my notation for the set difference. Some people prefer to write this as $S - T$.] And we showed in the last paragraph that $t \in T \implies t \not\sim x$, so $[x] = S \setminus T$.

Hence the equivalence classes are $\{S \setminus T, \{t\} \colon t \in T\}$, if $T \subsetneq S$ and $\{\{t\} \colon t \in T\}$ if $T = S$. So a complete set of equivalence class representatives can be obtained by taking T together with any element of $S \setminus T$, if $T \subsetneq S$, and by taking T if $T = S$.

(d) If $t_1, t_2 \in T$ then $(t_1\ t_2) \in H = G_{(T)}$, so $t_1 \sim t_2$. If $x \in S \setminus T$ then $t_1 \not\sim x$, since $\alpha(t_1) \in T$ for every $\alpha \in H$. So $[t_1] = T$.

Similarly, if $x, y \in S \setminus T$ then $[x] = S$. So provided $\emptyset \neq T \subsetneq S$, the equivalence classes are the elements of the set $\{T, S \setminus T\}$, and a complete set of equivalence class representatives is $\{t, x\}$ where t is any (fixed) element of T, and x is any (fixed) element of $S \setminus T$. If $T = \emptyset$ or $T = S$ then the collection of equivalence classes is $\{S\}$ and for any $k \in S$, the set $\{k\}$ is a complete set of equivalence class representatives.

5. Let S and T be non-empty sets and let $f \colon S \to T$ be a fixed mapping. Let $\approx$ be an equivalence relation on T, and consider the relation $\sim$ on S defined by

$$x \sim y \iff f(x) \approx f(y) \quad \text{for } x, y \in S.$$

(a) Prove that $\sim$ is an equivalence relation on S.

(b) Prove that $[x]_\sim = f^{-1}([f(x)]_\approx)$ for every $x \in S$.
[Notation explanation: if $U \subseteq T$ then $f^{-1}(U) = \{y \in S \colon f(y) \in U\}$].

(c) Explain how (a) may be applied to give alternative proofs that the two equivalence relations you identified in Exercise 3 really are equivalence relations.

Solution (a) $\sim$ **is reflexive** since $x \in S \implies f(x) \in T \implies f(x) \approx f(x) \iff x \sim x$.

$\sim$ **is symmetric** since for $x, y \in S$ we have $x \sim y \iff f(x) \approx f(y) \implies f(y) \approx f(x) \iff y \sim x$

$\sim$ **is transitive** since for $x, y, z \in S$ we have $x \sim y$ and $y \sim z \iff f(x) \approx f(y)$ and $f(y) \approx f(z) \implies f(x) \approx f(z) \iff x \sim z$.

(b) We have

$$\begin{aligned} f^{-1}([f(x)]_{\approx}) &= \{y \in S \colon f(y) \in [f(x)]_{\approx}\} \\ &= \{y \in S \colon f(y) \approx f(x)\} \\ &= \{y \in S \colon y \sim x\} \quad \text{by the definition of } \sim \\ &= [x]_{\sim}. \end{aligned}$$

(c) Let $f \colon P \to \mathbb{R}$, $\binom{x}{y} \mapsto y$ and consider the relation $\approx$ on $\mathbb{R}$ defined by $y \approx b \iff y - b \in \mathbb{Z}$. The relation $\sim$ in 3(a) is given by applying the definition in (a) to f and $\approx$, so we could prove that this is an equivalence relation by checking that $\approx$ is an equivalence relation on $\mathbb{R}$. [This is easy to do, but we omit it.]

Now take $f \colon P \to \mathbb{R}$, $p \mapsto d(p, 0)$ and let $\approx$ to be the equality relation $=$ on $\mathbb{R}$. This is certainly an equivalence relation, and the relation $\sim$ in 3(c) is given by applying the definition in (a) to f and $\approx$. Hence $\sim$ is an equivalence relation by (a).