Mathematics 1214: Introduction to Group Theory

Homework exercise sheet 5

Due 12:50pm, Friday 26th February 2010

1. Let $G = A_4$, so that

$$G = \{e, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3), (1\ 2\ 3), (1\ 3\ 2), (1\ 3\ 4), (1\ 4\ 3), (1\ 2\ 4), (1\ 4\ 2), (2\ 3\ 4), (2\ 4\ 3)\}$$

where e is the identity mapping on $\{1, 2, 3, 4\}$. Find G_T and $G_{(T)}$ for the following sets T:

(a) $T = \{1, 3\}$ (b) $T = \{2, 3, 4\}$ (c) $T = \{4\}$

Solution (a) A permutation in G is in G_T if and only if it fixes 1 and 3. But by checking each element of G in turn, we see that the only permutation in G fixing 1 and 3 is the identity mapping. So $G_T = \{e\}$. On the other hand, a permutation $\alpha \in G$ is in $G_{(T)}$ if and only if $\alpha(\{1, 3\}) = \{1, 3\} \iff \alpha\colon 1 \mapsto 1,\ 3 \mapsto 3$ or $\alpha\colon 1 \mapsto 3,\ 3 \mapsto 1$, and the only permutations in G with this property are e and $(1\ 3)(2\ 4)$. So $G_{(T)} = \{e, (1\ 3)(2\ 4)\}$.

(b) If $\alpha \in G_T$ then $\alpha(2) = 2$, $\alpha(3) = 3$ and $\alpha(4) = 4$. Since α is injective, this forces $\alpha(1) = 1$, and so $\alpha = e$, so $G_T = \{e\}$. If $\alpha \in G_{(T)}$ then $\alpha(T) = T$, and since α is injective, this again forces $\alpha(1) = 1$. So $\alpha(1) = 1$, and for $t \in T$, $\alpha(T)$ can be anything in T. So $G_{(T)} = G_{\{1\}} = \{e, (2\ 3\ 4), (2\ 4\ 3)\}$.

(c) A permutation in G is in G_T if and only if it fixes 4. So $G_T = \{e, (1\ 2\ 3), (1\ 3\ 2)\}$. Moreover, a permutation $\alpha \in G$ is in $G_{(T)}$ if and only if $\alpha(\{4\}) = \{4\} \iff \alpha(4) = 4 \iff \alpha \in G_T$. So $G_{(T)} = G_T$ in this case.

2. Let S be a set and let G be a subgroup of $(\mathrm{Sym}(S), \circ)$. Let A and B be subsets of S, let $H = G_A$ and let $K = G_{(A)}$. Prove that $G_{A\cup B} = H_B$, and give an example to show that $G_{(A\cup B)}$ need not be equal to $K_{(B)}$.

Solution We have $H = G_A = \{\alpha \in G\colon x \in A \implies \alpha(x) = x\}$. So

$$\begin{aligned} G_{A\cup B} &= \{\alpha \in G\colon x \in A \cup B \implies \alpha(x) = x\} \\ &= \{\alpha \in G\colon [x \in A \implies \alpha(x) = x] \text{ and } [x \in B \implies \alpha(x) = x]\} \\ &= \{\alpha \in G\colon \alpha \in H \text{ and } x \in B \implies \alpha(x) = x\} \\ &= \{\alpha \in H\colon x \in B \implies \alpha(x) = x\} \\ &= H_B. \end{aligned}$$

On the other hand, if $G = S_2$, $A = \{1\}$ and $B = \{2\}$ then $G_{(A\cup B)} = S_2$ and $K = \{e\}$, so $K_{(B)} = \{e\} \neq G_{(A\cup B)}$.

3. Let α, β be motions of the plane P. Show that

$$\alpha = \beta \iff \alpha(0) = \beta(0),\ \alpha(e_1) = \beta(e_1) \text{ and } \alpha(e_2) = \beta(e_2)$$

where, as usual, $0 = \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$ is the zero vector in P, $e_1 = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and $e_2 = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$.

Solution If $\alpha = \beta$ then $\alpha(p) = \beta(p)$ for every $p \in P$, so in particular, $\alpha(0) = \beta(0)$, $\alpha(e_1) = \beta(e_1)$ and $\alpha(e_2) = \beta(e_2)$.

Conversely, suppose that $\alpha(0) = \beta(0)$, $\alpha(e_1) = \beta(e_1)$ and $\alpha(e_2) = \beta(e_2)$. Let $\gamma = \beta^{-1} \circ \alpha$. We have $\gamma \in M$ (since M is a group under composition) and $\gamma(0) = 0$, $\gamma(e_1) = e_1$ and $\gamma(e_2) = e_2$. By Lemma 17, $\gamma = \iota_P$, the identity element of M. So $\beta^{-1} \circ \alpha = \iota_P$, and composing both sides on the left with β gives $\alpha = \beta \circ \beta^{-1} \circ \alpha = \beta \circ \iota_P = \beta$, so $\alpha = \beta$.

4. (a) Show that r, reflection in the x-axis, is a linear mapping $P \to P$.
   (b) Show that for any $\theta \in \mathbb{R}$, the mapping ρ_θ of rotation by θ counterclockwise about the origin, is a linear mapping $P \to P$.
   (c) For which $a \in P$ is the mapping τ_a of translation by a a linear mapping?

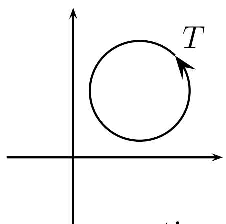
   [Recall that $\alpha\colon P \to P$ is linear if $p, q \in P \implies \alpha(p+q) = \alpha(p) + \alpha(q)$ and $p \in P$, $\lambda \in \mathbb{R} \implies \alpha(\lambda p) = \lambda\alpha(p)$.]

Solution (a) If $p = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in P$ then $r(p) = \left(\begin{smallmatrix} x \\ -y \end{smallmatrix}\right) = Rp$ where $R = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. So for $p, q \in P$ we have $r(p+q) = R(p+q) = Rp + Rq = r(p) + r(q)$ and for $p \in P$, $\lambda \in \mathbb{R}$ we have $r(\lambda p) = R(\lambda p) = \lambda Rp = \lambda r(p)$, using properties of matrix multiplication. So r is linear.

(b) If $p \in P$ then $\rho_\theta(p) = Sp$ where $S = \left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)$. Replacing R with S and r with ρ_θ in the argument of the previous paragraph shows that ρ_θ is linear.

(c) Any linear mapping maps the zero vector to itself. So if τ_a is linear, then $0 = \tau_a(0) = 0 + a = a$, so $a = 0$. Moreover, $\tau_0 = \iota_P$ is the identity mapping, which is certainly linear. So the answer is: τ_a is linear if and only if $a = 0$.

5. Let P be the plane $P = \mathbb{R}^2$ and let M be the set of motions of P. We say that a motion $\alpha \in M$ is *proper* if $\alpha = \tau_a \circ \rho_\theta$ for some $a \in P$ and $\theta \in \mathbb{R}$, and we say that α is *improper* if $\alpha = \tau_a \circ \rho_\theta \circ r$ for some $a \in P$ and $\theta \in \mathbb{R}$.
   (a) Explain why every motion is either proper or improper.
   (b) If C is a circle in the plane and $\alpha \in M$, show that $\alpha(C)$ is a circle in the plane.
   (c) Let T be a circle in the plane P together with an arrow in the counterclockwise direction, as shown. Give a geometric argument to show that if α is a motion, then α is a proper motion if and only if $\alpha(T)$ is a circle with an arrow in the counterclockwise direction. Give a similar statement for improper motions and explain why it is correct.



   [The arrow indicates the *orientation* of the circle. Thus the proper motions are the orientation preserving motions.]
   (d) Fill in the last column of the following table with "proper" or "improper", and explain why your answers are (always) correct.

α	β	$\alpha \circ \beta$
proper	proper	
proper	improper	
improper	proper	
improper	improper	

Solution (a) This is what Theorem 18 says.

(b) Let q be the centre of the circle C and let s be the radius of C. Then $C = \{p \in P\colon d(p,q) = s\}$, so

$$\begin{aligned}\alpha(C) &= \{\alpha(p)\colon d(p,q) = s\}\\ &= \{r \in P\colon r = \alpha(p),\ p \in P,\ d(p,q) = s\}\\ &= \{r \in P\colon d(\alpha^{-1}(r), q) = s\}\\ &= \{r \in P\colon d(r, \alpha(q)) = s\} \quad \text{since } \alpha \in M \text{ and } \alpha(\alpha^{-1}(r)) = r.\end{aligned}$$

Hence $\alpha(C)$ is the circle centred at $\alpha(q)$ of radius s.

(b) First, let's check this for the special cases $\alpha = \rho_\theta$ and $\alpha = \tau_a$.

(i) Clearly [draw a picture!] $\rho_\theta(T)$ is a circle with an arrow pointing counterclockwise.

(ii) Clearly [draw a picture!] $\tau_a(T)$ is a circle with an arrow pointing counterclockwise.

(iii) And, ever so slightly less clearly, but still pretty clearly [draw another picture!] $r(T)$ is a circle with an arrow pointing clockwise.

Now if α is a proper motion then $\alpha = \tau_a \circ \rho_\theta$ for some a, θ, so $\alpha(T) = \tau_a(\rho_\theta(T))$. Since $T' = \rho_\theta(T)$ is a circle with counterclockwise arrow by (i), $\alpha(T) = \tau_a(T')$ is a circle with counterclockwise arrow by (ii).

If S is a circle with an arrow pointing clockwise, then

(i)' Clearly [draw a picture!] $\rho_\theta(S)$ is a circle with an arrow pointing clockwise.

(ii)' Clearly [draw a picture!] $\tau_a(S)$ is a circle with an arrow pointing clockwise.

If α is not a proper motion, then it is improper, so $\alpha = \tau_a \circ \rho_\theta \circ r$ so $\alpha(T) = \tau_a(\rho_\theta(r(T)))$. Let $S = r(T)$. Then S is a circle with clockwise arrow by (iii), so by (i)' and (ii)', $\alpha(T) = \tau_a(\rho_\theta(S))$ is a circle with clockwise arrow.

So we have shown:

$$\alpha \text{ proper} \implies \alpha(T) \text{ is a counterclockwise circle}$$

and

$$\alpha \text{ not proper} \implies \alpha(T) \text{ is not a counterclockwise circle.}$$

So α is proper $\iff$ $\alpha(T)$ is a counterclockwise circle.

Hence α is improper $\iff$ $\alpha(T)$ is not a counterclockwise circle. But $\alpha(T)$ is a circle for every motion M by (b), so if it is not a counterclockwise circle then it must be a clockwise circle. So α is improper $\iff$ $\alpha(T)$ is a clockwise circle.

(d) The last column reads "proper, improper, improper, proper". The reason is that the proper motions are those that preserve orientation of circles, and the improper motions are those that reverse orientations, by (c). So the composition of an even number of improper motions with any number of proper motions preserves orientations, so is proper, whereas the composition of an odd number of improper motions with any number of proper motions reverses orientations, so is improper.