Mathematics 1214: Introduction to Group Theory

Homework exercise sheet 4

Due 12:50pm, Friday 19th February 2010

1. (a) Show that $H = \{(1), (1\ 3\ 4), (1\ 4\ 3)\}$ is a subgroup of $(S_4, \circ)$.
   [Here, (1) represents the identity permutation in S_4].
   (b) Show that $K = \{(1), (1\ 2\ 3), (1\ 3\ 2), (1\ 3)\}$ is not a subgroup of $(S_4, \circ)$.
   (c) Show that $L = \{\alpha \in S_4 \colon \alpha(1) = 1\}$ is a non-abelian subgroup of $(S_4, \circ)$.
   [This means: show that L is a subgroup of $(S_4, \circ)$, and that the group $(L, \circ)$ is not abelian.]

Solution (a) Each element of H is in S_4 and H is clearly not empty. Consider the part of Cayley table for $\circ$ corresponding to elements of H:

$\circ$	(1)	$(1\ 3\ 4)$	$(1\ 4\ 3)$
(1)	(1)	$(1\ 3\ 4)$	$(1\ 4\ 3)$
$(1\ 3\ 4)$	$(1\ 3\ 4)$	$(1\ 4\ 3)$	(1)
$(1\ 4\ 3)$	$(1\ 4\ 3)$	(1)	$(1\ 3\ 4)$

Examining this, we see that for all $x, y \in H$ we have $x \circ y \in H$, which is the same as saying that $x, y \in H \implies x \circ y \in H$. Moreover, since $(1)^{-1} = (1) \in H$ and $(1\ 3\ 4)^{-1} = (1\ 4\ 3) \in H$ and $(1\ 4\ 3)^{-1} = (1\ 3\ 4) \in H$, we have $x \in H \implies x^{-1} \in H$. So H is a subgroup of $(S_4, \circ)$.

(b) For example, let

(b) We have $(1\ 2\ 3) \in K$ and $(1\ 3) \in K$, but $(1\ 2\ 3) \circ (1\ 3) = (2\ 3) \notin K$. So $x, y \in K \implies x \circ y \in K$ is false. So K is not a subgroup of $(S_4, \circ)$.

(c) If we write $G = S_4$ then $L = G_{\{1\}}$. So L is a subgroup of G, by a theorem proven in class. Now $(2\ 3\ 4) \in L$ and $(2\ 3) \in L$, but $(2\ 3\ 4) \circ (2\ 3) = (2\ 4) \neq (3\ 4) = (2\ 3) \circ (2\ 3\ 4)$. So $\circ$ is not commutative on L, so $(L, \circ)$ is not an abelian group.

2. (a) Is $\{A \in GL(2, \mathbb{R}) \colon \det(A) > 0\}$ a subgroup of $(GL(2, \mathbb{R}), \text{matrix multiplication})$?
   (b) Is $\{A \in GL(2, \mathbb{R}) \colon \det(A) < 0\}$ a subgroup of $(GL(2, \mathbb{R}), \text{matrix multiplication})$?

Solution (a) Yes, $H = \{A \in GL(2, \mathbb{R}) \colon \det(A) > 0\}$ is a subgroup of $GL(2, \mathbb{R})$. Indeed:

- $H \subseteq GL(2, \mathbb{R})$ and $I \in H$, so $H \neq \emptyset$
- $A, B \in H \implies \det(A) > 0$ and $\det(B) > 0 \implies \det(AB) = \det(A)\det(B) > 0 \implies AB \in H$
- if $A \in H \implies \det(A) > 0 \implies \det(A^{-1}) = \frac{1}{\det(A)} > 0 \implies A^{-1} \in H$.

(b) No, $K = \{A \in GL(2,\mathbb{R})\colon \det(A) < 0\}$ is not a subgroup of $GL(2,\mathbb{R})$ since, for example, the matrix $A = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ has determinant -1, so $A \in K$, but $A \cdot A = I \notin K$. So K is not closed under matrix multiplication, so K is not a subgroup of $(GL(2,\mathbb{R}), \text{matrix multiplication})$.

3. Recall that if $+$ denotes vector addition, then $(\mathbb{R}^2, +)$ is a group with identity element $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, and the inverse of a vector $\mathbf{v} \in \mathbb{R}^2$ is $-\mathbf{v}$.

A *subspace* of $\mathbb{R}^2$ is a non-empty set $V \subseteq \mathbb{R}^2$ such that

$$\mathbf{v}, \mathbf{w} \in V,\ \lambda \in \mathbb{R} \implies \mathbf{v} + \lambda\mathbf{w} \in V.$$

(a) Prove that if V is a subspace of $\mathbb{R}^2$, then V is a subgroup of $(\mathbb{R}^2, +)$.

(b) Find an example of a subgroup of $(\mathbb{R}^2, +)$ which is not a subspace of $\mathbb{R}^2$.

Solution (a) Suppose that V is a subspace of $\mathbb{R}^2$. Then $V \neq \emptyset$, by the definition of a subspace. Also, if $\mathbf{v}, \mathbf{w} \in V$ then $\mathbf{v} + \mathbf{w}^{-1} = \mathbf{v} - \mathbf{w} = \mathbf{v} + (-1)\mathbf{w} \in V$, using the definition of a subspace (with $\lambda = -1$). Hence by Theorem 11, V is a subgroup of $(\mathbb{R}^2, +)$.

(b) For example, let $H = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x, y \in \mathbb{Z} \right\}$. Clearly, $H \neq \emptyset$, and if $\mathbf{v}, \mathbf{w} \in H$ then $\mathbf{v} + \mathbf{w} \in H$ and $\mathbf{v}^{-1} = -\mathbf{v} \in H$, so H is a subgroup of $(\mathbb{R}^2, +)$. However, $\mathbf{0} \in H$ and $\mathbf{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in H$ but $\mathbf{0} + \frac{1}{2}\mathbf{w} \notin H$, so H is not a subspace of $\mathbb{R}^2$.

4. Prove that if H_1 and H_2 are subgroups of a group $(G, *)$, then $H_1 \cap H_2$ is a subgroup of $(G, *)$.

Solution Since $H_1 \subseteq G$ and $H_2 \subseteq G$, we have $H_1 \cap H_2 \subseteq G$.

Let e be the identity element of $(G, *)$. Since H_1 and H_2 are subgroups of G, we have $e \in H_1$ and $e \in H_2$, by Theorem 9. So $e \in H_1 \cap H_2$. So $H_1 \cap H_2 \neq \emptyset$.

We have $x, y \in H_1 \implies x * y \in H_1$ and $x, y \in H_2 \implies x * y \in H_2$. So $x, y \in H_1 \cap H_2 \implies x, y \in H_1$ and $x, y \in H_2 \implies x * y \in H_1$ and $x * y \in H_2 \implies x * y \in H_1 \cap H_2$.

We have $x \in H_1 \implies x^{-1} \in H_1$ and $x \in H_2 \implies x^{-1} \in H_2$. So $x \in H_1 \cap H_2 \implies x \in H_1$ and $x \in H_2 \implies x^{-1} \in H_1$ and $x^{-1} \in H_2 \implies x^{-1} \in H_1 \cap H_2$.

Hence $H_1 \cap H_2$ is a subgroup of $(G, *)$.

5. Prove that the following statement is *false*:

If H_1 and H_2 are subgroups of a group $(G, *)$, then $H_1 \cup H_2$ is a subgroup of $(G, *)$.

[Hint: you should find a counterexample. This means: find a group $(G, *)$ and subgroups H_1 and H_2 such that the conclusion of the statement above is not true. If you're stuck for ideas, you could try looking at subgroups of $(G, *) = (S_3, \circ)$.]

Solution For example, if $(G, *) = (S_3, \circ)$ then $H_1 = G_{\{1\}} = \{(1),\ (2\ 3)\}$ and $H_2 = G_{\{2\}} = \{(1),\ (1\,3)\}$ are both subgroups of G by Theorem 13. However, $H = H_1 \cup H_2 = \{(1),\ (2\,3),\ (1\,3)\}$ is not a subgroup of $(S_3, \circ)$, since $(2\ 3) \in H$ and $(1\ 3) \in H$ but $(2\ 3)(1\ 3) = (1\ 2\ 3) \notin H$.

6. Let $(G, *)$ be a group.

   (a) If $x, y \in G$, show that $(x * y * x^{-1})^{-1} = x * y^{-1} * x^{-1}$.

   (b) Suppose that H is a subgroup of $(G, *)$. Let $x \in G$ and consider the set

   $$K = \{x * y * x^{-1} \colon y \in H\}.$$

   Prove that K is a subgroup of $(G, *)$.

Solution (a) Let $w = x * y * x^{-1}$ and $z = x * y^{-1} * x^{-1}$. Using associativity several times, we have

$$\begin{aligned} w * z &= (x * y * x^{-1}) * (x * y^{-1} * x^{-1}) = x * y * (x^{-1} * x) * y^{-1} * x^{-1} \\ &= x * y * e * y^{-1} * x^{-1} = x * (y * y^{-1}) * x^{-1} = x * e * x^{-1} = x * x^{-1} = e \end{aligned}$$

so $w * z = e$, and interchanging y and y^{-1} in this calculation gives $z * w = e$. So $z = w^{-1}$.

(b) Since G is closed under $*$, we have $K \subseteq G$, and $e \in H \implies x * e * x^{-1} \in K$, so $K \neq \emptyset$. If $a, b \in K$ then $a = x * y * x^{-1}$, $b = x * z * x^{-1}$ for some $y, z, \in H$. Since H is a subgroup, we have $y * z \in H$ and $y^{-1} \in H$, so

$$a * b = (x * y * x^{-1}) * (x * z * x^{-1}) = x * (y * z) * x^{-1} \in K$$

and

$$a^{-1} = (x * y * x^{-1})^{-1} = x * y^{-1} * x^{-1} \in K.$$

So $a, b \in K \implies a * b \in K$ and $a \in K \implies a^{-1} \in K$. So K is a subgroup of $(G, *)$.